INTERMEDIATE
ALGEBRA

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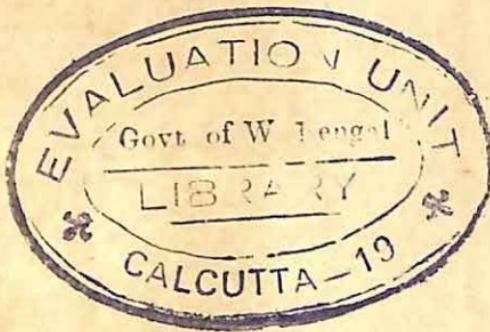
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BY

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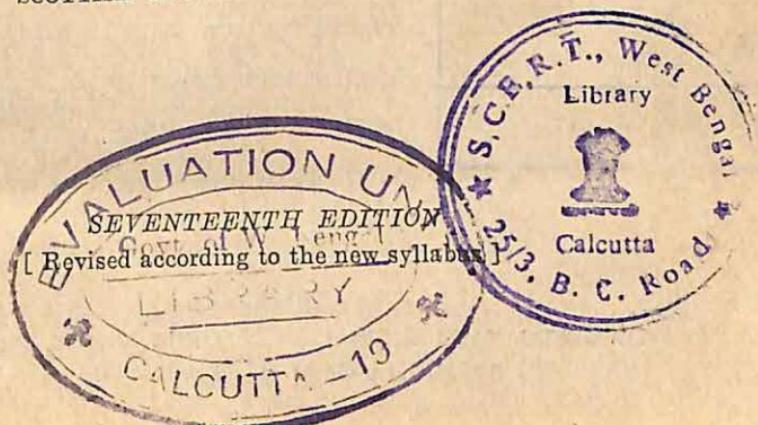
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PREFACE TO THE FIRST EDITION

The present work is designed to be a text-book for the use of students preparing for the Intermediate Examinations of the Indian Universites, especially the University of Calcutta. No pains have been spared to make the exposition clear and concise, without going into unnecessary details but at the same time making the treatment as much rigorous and up-to-date as may be possible within the scope of an elementary work.

Important formulæ and results have been inserted in the beginning of the book for ready reference. A good number of typical examples have been worked out by way of illustrations and attempts have been made to ensure variety and interest in the examples for exercise, which include many set at the University Examinations. Calcutta University questions for the Intermediate Examinations of recent years are inserted at the end of the book with a view to given an idea of the standard demanded of the student.

Our thanks are due to several of our friends and colleagues—Prof. B. C. Das, M.Sc. of the Presidency College, Prof. N. Karforma, M. Sc. of the Scottish Church College, Prof. P. C. Bhattacharya, M. A. of the Bangabasi College, Prof. S. K. Chatterjee, M. Sc. of the St. Xavier's College, Prof. M. Roy Choudhury, M. A. of the Presidency College, for their helpful suggestions in the preparation of the work and also to Prof. H. K. Ganguli, M. A. of the Scottish Church College for help in the verification of the answers of the examples.

Any correction or suggestions for the improvement of the work will be thankfully received.

CALCUTTA :
June, 1938   THE AUTHORS

PREFACE TO THE FIFTEENTH EDITION

This edition has been revised according to the new syllabus of the Calcutta University. Topics included in the new syllabus have been inserted in the proper places and in the two Appendices.

Our best thanks are due to our pupils Prof. Jyoti Choudhury, M. Sc. of Lady Brabourne College and Sri. Tapen Moulik, M. Sc. for detecting certain misprints and helping us considerably in bringing out this edition in proper time.

Our thanks are also due to the authorities and the staff of Messrs K. P. Basu Printing Works for the efficient discharge of their duties, in spite of their various preoccupations.

CALCUTTA :
1st September, 1959   THE AUTHORS

PREFACE TO THE SIXTEENTH EDITION

This edition is practically a reprint of the previous edition. The text has been thoroughly revised. Our best thanks are due to our pupils Prof. Tapen Moulik, M.Sc. of the B. E. College, and Miss Kalpana Sarkar, B.Sc. for helping us considerably in bringing out this edition in proper time.

CALCUTTA :
1st July, 1962   THE AUTHORS

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NEW SYLLABUS OF ALGEBRA

for

Intermediate Examinations of the Calcutta University

from 1958

1. Logarithms.

2. Complex Numbers—Geometrical representation of a Complex number.

3. Quadratic Equations.

4. Simultaneous equations in two unknowns, one of which is quadratic and the other linear.

5. Theory of Quadratic Equations and Expressions.

6. Permutation and Combination.

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12. Use of the Series for e^x and $\log_e(1+x)$, and their direct application (the proof of the expansions are not required).

13. Acquaintance with the graphs of x^n (n a positive integer), $\log x$ and e^x.

IMPORTANT FORMULÆ AND RESULTS

1. (i) If $\dfrac{a}{b} = \dfrac{c}{d}$, then each $= \dfrac{a+c}{b+d} = \dfrac{a-c}{b-d}$.

(ii) If $\dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.

(iii) If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \cdots\cdots$,

then each $= \dfrac{\text{sum of all numerators}}{\text{sum of all denominators}}$

$$= \dfrac{pa + qc + re + \cdots}{pb + qd + rf + \cdots}$$

2. If $a + \sqrt{b} = c + \sqrt{d}$, then $a = c$ and $b = d$.

3. If $a + ib = c + id$, then $a = c$ and $b = d$.

4. $ax^2 + bx + c = 0$;

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$a + \beta = -\frac{b}{a} \; ; \; a\beta = \frac{c}{a}.$$

5. $^nP_r = n(n-1)(n-2)\cdots\cdots(n-r+1)$

$$= \frac{\lfloor n}{\lfloor n-r}$$

$^nP_n = \lfloor n.$

Permutations of things not all different $= \dfrac{\lfloor n}{\lfloor p \lfloor q \lfloor r}$.

6. $^nC_r = \dfrac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r}$

$$= \frac{\lfloor n}{\lfloor r \; \lfloor n-r}$$

$$^nC_r = {}^nC_{n-r}$$

$$^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$$

$$^nC_r = \frac{n-r+1}{r} \times {}^nC_{r-1} = \frac{n}{r} \cdot {}^{n-1}C_{r-1}$$

$$^nC_1 + {}^nC_2 + \cdots\cdots + {}^nC_n = 2^n - 1.$$

7. (i) When n is a *positive integer,*

$$(a+x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \cdots\cdots + x^n$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{\lfloor 2} a^{n-2}x^2 + \cdots\cdots + x^n$$

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots\cdots + x^n$$

$$= 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \cdots\cdots + x^n.$$

$(r+1)$th term in the expansion of $(a+x)^n$

$$= {}^nC_r a^{n-r}x^r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} a^{n-r}x^r.$$

$(r+1)$th term in the expansion of $(1+x)^n$

$$= {}^nC_r x^r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} x^r.$$

(ii) When n is *fractional or negative,*

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \cdots,$$

provided x is numerically less than unity.

$(r+1)$th term in the expansion of $(1+x)^n$

$$= \frac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{\lfloor r} x^r.$$

INTERMEDIATE ALGEBRA

RATIO AND PROPORTION

1. Recapitulation. The following important result on ratios and proportions are quoted here for ready reference :

(i) Commensurable and Incommensurable Quantities.

Two quantities are said to be *commensurable*, if their ratio can be expressed *exactly* as the ratio of two integers, otherwise they are said to be *incommensurable*. Since $1\frac{3}{4} : 2\frac{2}{3} = 21 : 32$, $1\frac{3}{4}$ and $2\frac{2}{3}$ are two commensurable quantities.

A single quantity is said to be incommensurable when it is incommensurable with unity. Thus $\sqrt{2}$, $\sqrt{3}$ are incommensurable quantities.

(ii) Continued Proportion.

Quantities are said to be in *continued proportion* when the first is to the second, as the second is to the third, as the third is to the fourth ; and so on. Thus, $a, b, c, d, \ldots$ are in continued proportion when $a : b = b : c = c : d = \cdots$.

If three quantities a, b, c be in continued proportion, b is called the *mean proportional between* a and c.

(iii) Componendo and dividendo.

If $\quad \dfrac{a}{b} = \dfrac{c}{d}$, then $\dfrac{a+b}{a-b} = \dfrac{c+d}{c-d}$.

Let $\quad \dfrac{a}{b} = \dfrac{c}{d} = k$ then $a = bk$, $c = dk$.

$\therefore \dfrac{a+b}{a-b} = \dfrac{bk+b}{bk-b} = \dfrac{k+1}{k-1}$; $\dfrac{c+d}{c-d} = \dfrac{dk+d}{dk-d} = \dfrac{k+1}{k-1}$,

whence, the result follows easily.

(iv) Two Important Theorems.

(A) If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \cdots$, then

$$\text{each ratio} = \left[\frac{pa^n + qc^n + re^n + \cdots}{pb^n + qd^n + rf^n + \cdots}\right]^{\frac{1}{n}}$$

where p, q r,... are any quantities whatever.

Let $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \cdots = k$.

Then, $a = bk,\ c = dk,\ e = fk,\ldots$

$\therefore\quad pa^n = pb^n k^n,\ qc^n = qd^n k^n,\ re^n = rf^n k^n,\ldots$

$\therefore\quad \dfrac{pa^n + qc^n + re^n + \cdots}{pb^n + qd^n + rf^n + \cdots} = k^n.$

Taking the nth root of both sides,

$$\left[\frac{pa^n + qc^n + re^n + \cdots}{pb^n + qd^n + rf^n + \cdots}\right]^{\frac{1}{n}} = k = \frac{a}{b} = \frac{c}{d} = \text{etc.}$$

Cor. If $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \cdots$, then,

(a) each ratio $= \dfrac{a + c + e + \cdots}{b + d + f + \cdots} = \dfrac{\text{sum of all the numerators}}{\text{sum of all the denominators}}$,

(b) each ratio $= \dfrac{pa + qc + re + \cdots}{pb + qd + rf + \cdots}.$

(B) If $\dfrac{a}{b}$, $\dfrac{c}{d}$, $\dfrac{e}{f}$, $\cdots \dfrac{p}{q}$ be a number of unequal ratios of

which the denominators are all positive, then

$$\frac{a + c + e + \cdots + p}{b + d + f + \cdots + q}$$

lies between the greatest and the least of them.

Let $\dfrac{a}{b}$ be the greatest of the ratios and let it be denoted

by k; then $a = bk$.

Since $\dfrac{c}{d}$, $\dfrac{e}{f} \cdots < k$. $\therefore$ $c < dk$, $e < fk$,...

$\therefore$ $(a + c + e + \cdots + p) < k\,(b + d + f + \cdots + q)$,

since the denominators are all positive.

$\therefore$ $\dfrac{a + c + e + \cdots + p}{b + d + f + \cdots + q} < k$, $i.e.,\ < \dfrac{a}{b}$, greatest of the

ratios.

Again suppose $\dfrac{p}{q}$ is the least of the ratios and let it be

denoted to k''; $\therefore$ $p = k'q$.

Since $\dfrac{a}{b}$, $\dfrac{c}{d}$, $\cdots > k'$,

$\therefore$ $a > bk'$, $c > dk'$,...

$\therefore$ $(a + c + e + \cdots + p) > k'\,(b + d + f + \cdots + q)$,

$\therefore$ $\dfrac{a + c + e + \cdots + p}{b + d + f + \cdots + q} > k'$ $i.e.,\ > \dfrac{p}{q}$, least of the ratios.

2. Illustrative Examples.

Ex. 1. *If a_1, a_2,.........a_n be in continued proportion, show that*

$$\frac{a_1}{a_n} = \left(\frac{a_1}{a_2}\right)^{n-1}.$$ [*C. U. 1925*]

We have $\dfrac{a_1}{a_2} = \dfrac{a_2}{a_3} = \dfrac{a_3}{a_4} = \cdots = \dfrac{a_{n-1}}{a_n} = k$ (suppose) ;

$\therefore$ $k^{n-1} = \dfrac{a_1}{a_2} \times \dfrac{a_2}{a_3} \cdots \times \dfrac{a_{n-1}}{a_n} = \dfrac{a_1}{a_n}.$

Also, $k^{n-1} = \left(\dfrac{a_1}{a_2}\right)^{n-1}$; $\therefore$ $\dfrac{a_1}{a_n} = \left(\dfrac{a_1}{a_2}\right)^{n-1}.$

Ex. 2. *If $x\,(b - c) + y\,(c - a) + z\,(a - b) = 0$, then*

(i) $\dfrac{b - c}{y - z} = \dfrac{c - a}{z - x} = \dfrac{a - b}{x - y}.$ (ii) $\dfrac{b - c}{bz - cy} = \dfrac{c - a}{cx - az} = \dfrac{a - b}{ay - bx}.$

We have $x(b - c) + y\,(c - a) + z\,(a - b) = 0$ $\cdots$ $\cdots$ (1)

also, identically $(b - c) + (c - a) + (a - b) = 0$ $\cdots$ $\cdots$ (2)

and $a\,(b - c) + b\,(c - a) + c\,(a - b) = 0.$ $\cdots$ $\cdots$ (3)

From (1) and (2), by the Rule of Cross Multiplication, we get the first result, and from (3) and (1) by the Rule of Cross Multiplication, we get the second result.

Examples I

1. (a) If $a(y + z) = b(z + x) = c(x + y)$, show that

$$\frac{y - z}{a(b - c)} = \frac{z - x}{b(c - a)} = \frac{x - y}{c(a - b)}.$$

(b) If $(m - p) : (n - p)$ is equal to $(m + q) : (n + q)$, show that $p + q = 0$, or, $m = n$.

2. If $\dfrac{a}{b + c - a} = \dfrac{b}{c + a - b} = \dfrac{c}{a + b - c}$, prove that each ratio $= 1$, and $a = b = c$ if $a + b + c \neq 0$.

3. If $\dfrac{x}{a^3} = \dfrac{y}{b^2} = \dfrac{z}{c^2}$, then

$$\frac{x + y + z}{a^2 + b^2 + c^2} = \frac{xa^{-1} + yb^{-1} + zc^{-1}}{a + b + c}.$$

4. If $\dfrac{l - m}{ly + mx} = \dfrac{m - n}{mz + lx} = \dfrac{n - l}{nx + lz} = \dfrac{l + m + n}{lx + my + nz}$, then

each of these ratios $= \dfrac{1}{x + y + z}$ if $l + m + n \neq 0$.

5. If $\dfrac{a}{y + z} = \dfrac{b}{z + x} = \dfrac{c}{x + y}$, prove that

$$\frac{a(a - c)}{y^2 - z^2} = \frac{b(c - a)}{z^2 - x^2} = \frac{c(a - b)}{x^2 - y^2}.$$

6. If $\dfrac{x^2 - yz}{a} = \dfrac{y^2 - zx}{b} = \dfrac{z^2 - xy}{c}$, then

$$(x + y + z)(a + b + c) = ax + by + cz.$$

7. If $\dfrac{x_1}{x_2} = \dfrac{x_2}{x_3} = \dfrac{x_3}{x_4} = \cdots = \dfrac{x_n}{x_{n+1}}$, prove that

$$\left(\dfrac{x_1 + x_2 + \cdots + x_n}{x_2 + x_3 + \cdots + x_{n+1}} \right)^n = \dfrac{x_1}{x_{n+1}}.$$

8. (a) If $\dfrac{bz + cy}{b - c} = \dfrac{cx + az}{c - a} = \dfrac{ay + bx}{a - b}$, show that

$$(a + b + c)(x + y + z) = ax + by + cz.$$

(b) If $p : q :: r : s$, show that

$$\left(\dfrac{1}{p} + \dfrac{1}{s} \right) - \left(\dfrac{1}{q} + \dfrac{1}{r} \right) = \dfrac{(p - q)(p - r)}{pqr}.$$

9. If $(a^2 + b^2 + c^2)(x^2 + y^2 + z^2) = (ax + by + cz)^2$,

show that $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$.

10. If $a(by + cz - ax) = b(cz + ax - by) = c(ax + by - cz)$

then (i) $\dfrac{y + z - x}{a} = \dfrac{z + x - y}{b} = \dfrac{x + y - z}{c}$,

(ii) $\dfrac{x}{b + c} = \dfrac{y}{c + a} = \dfrac{z}{a + b}$.

11. If the work done by $x - 1$ men in $x + 1$ days is to the work done by $x + 1$ men in $x + 2$ days be in the ratio of $5 : 6$, find x.

12. If four positive quantities are in continued proportion, show that the difference between the first and last is at least three times as great as the difference between the other two.

13. In a certain examination, the number of successful candidates was 3 times the number of unsuccessful candidates. If there had been 16 fewer candidates and if 16 more would have been unsuccessful, the numbers would have been as 2 to 1. Find the number of candidates.

14. If $a_1, a_2, \ldots a_n$ be in continued proportion, then
$$a_2 = {}^{n-1}\!\!\sqrt{(a_1{}^{n-2}a_n)} \;;\; a_3 = {}^{n-1}\!\!\sqrt{(a_1{}^{n-3}a_n{}^2)} \;;\; \cdots\cdots$$
$$a_r = {}^{n-1}\!\!\sqrt{(a_1{}^{n-r}a_n{}^{r-1})}.$$

15. If $\sqrt{(x-a)^2 + (y-b)^2} = \sqrt{x^2+y^2} - \sqrt{a^2+b^2}$, then
$$x : y = a : b.$$

16. (i) If $\dfrac{a}{b+c} = \dfrac{b}{c+a} = \dfrac{c}{a+b}$, prove that each of these fractions is equal to $\frac{1}{2}$ or -1.

(ii) If $a+b : b+c = c+d : d+a$, then either
$$c = a \text{ or } a+b+c+d = 0.$$

17. If $\dfrac{x}{b+c-a} = \dfrac{x}{c+a-b} = \dfrac{z}{a+b-c}$, then
$$(a+b+c)(yz + zx + xy) = (x+y+z)(ax + by + cz).$$

18. If $\dfrac{3x+2y}{3a-2b} = \dfrac{3y+2z}{3b-2c} = \dfrac{3z+2x}{3c-2a}$, then
$$5(x+y+z)(5c+4b-3a) = (9x+8y+13z)(a+b+c).$$

19. If $\dfrac{2y+2z-x}{a} = \dfrac{2z+2x-y}{b} = \dfrac{2x+2y-z}{c}$, then
$$\frac{x}{2b+2c-a} = \frac{y}{2c+2a-b} = \frac{z}{2a+2b-c}.$$

20. (i) If $\dfrac{by+cz}{b^2+c^2} = \dfrac{cz+ax}{c^2+a^2} = \dfrac{ax+by}{a^2+b^2}$, prove that
$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c}.$$

(ii) If $\dfrac{bz+cy}{a} = \dfrac{cx+az}{b} = \dfrac{ay+bx}{c}$, prove that
$$\frac{x}{a(b^2+c^2-a^2)} = \frac{y}{b(c^2+a^2-b^2)} = \frac{z}{c(a^2+b^2-c^2)}.$$

21. If $\dfrac{x}{l\,(mb + nc - la)} = \dfrac{y}{m\,(nc + la - mb)} = \dfrac{z}{n\,(la + mb - nc)},$

prove that

$$\frac{l}{x\,(by + cz - ax)} = \frac{m}{y\,(cz + ax - by)} = \frac{n}{z\,(ax + by - cz)}.$$

22. (i) If $a\,(y + z) = x,\ b\,(z + x) = y,\ c\,(x + y) = z,$ then

$$\frac{x^2}{a\,(1 - bc)} = \frac{y^2}{b\,(1 - ca)} = \frac{z^2}{c\,(1 - ab)}.$$

(ii) If $\dfrac{y}{z} - \dfrac{z}{y} = \dfrac{b - c}{x},\ \dfrac{z}{x} - \dfrac{x}{z} = \dfrac{c - a}{y},$

prove that $\quad \dfrac{x}{y} - \dfrac{y}{x} = \dfrac{a - b}{z}.$

23. If $\dfrac{a^2 - bc}{x} = \dfrac{b^2 - ca}{y} = \dfrac{c^2 - ab}{z},$ then

(i) $x^2 - yz : y^2 - zx : z^2 - xy = a : b : c$;

(ii) $x - y : y - z ; z - x = a - b : b - c : c - a.$

24. If $\dfrac{x}{b - c} + \dfrac{y}{c - a} + \dfrac{z}{a - b} = 0$ and

$$\frac{x}{b + c} + \frac{y}{c + a} + \frac{z}{a + b} = 0, \text{ then}$$

$$\frac{x}{(b^2 - c^2)(a^2 - bc)} = \frac{y}{(c^2 - a^2)(b^2 - ca)} = \frac{z}{(a^2 - b^2)(c^2 - ab)}.$$

25. (i) If $x = cy + bz,\ y = az + cx,\ z = bx + ay,$ show that

$$\frac{x^2}{1 - a^2} = \frac{y^2}{1 - b^2} = \frac{z^2}{1 - c^2}.$$

(ii) If $ax + hy + gz = 0,\ hx + by + fz = 0,\ gx + fy + cz = 0,$

then $\dfrac{x^2}{bc - f^2} = \dfrac{y^2}{ca - g^2} = \dfrac{z^2}{ab - h^2}.$

ANSWERS

13. 136.

11. 16.

3. Direct Variation.

A quantity is said to *vary directly* as another, if the two are so related that when one is changed, the other is also changed in the same proportion.

The circumference of a circle $= 2\pi r$; hence the circumference of a circle varies directly as the radius, for if the radius be doubled, trebled etc., the circumference is also doubled, trebled etc. Similarly, the area of a triangle of given base varies directly as the altitude; the fee-income of a college varies directly as the number of students.

The word '*directly*' is generally omitted for the sake of brevity, so when we say 'A varies as B' we mean 'A varies directly as B' and the notation used to denote this is $A \propto B$, the symbol $\propto$ denoting '*varies as*'.

From the above definition, it follows that

If A varies as B, then $A = mB$, where m is constant.

Let A be changed to a and in consequence let B be change to b, then by the above definition, we have

$$\frac{A}{a} = \frac{B}{b}, \text{ or, } \frac{A}{B} = \frac{a}{b}; \quad \therefore \quad A = \frac{a}{b} \cdot B = mB;$$

where m is equal to the constant quantity $\frac{a}{b}$.

This constant m is called *the constant of variation*.

The constant of variation can be determined if any pair of the corresponding values of the variables are given.

Suppose it is given, '$A \propto B$ and $A = 15$, when $B = 3$.'

The constant of variation can be determined thus :

$$\because \quad A \propto B, \quad \therefore \ A = mB \ ; \ \therefore \ 15 = m.3 \ ; \ \therefore \ m = 5.$$

Conversely, *if $A = mB$, where m is a constant*, then $A \propto B$ because if B is changed, A is also changed in the same proportion.

4. Inverse Variation.

One quantity is said to vary *inversely* as another, when the first varies directly as the reciprocal of the second.

Thus, A varies inversely as B, if $A \propto \dfrac{1}{B}$, *i.e.*, if $A = m \cdot \dfrac{1}{B}$,

or, $AB = m$, where m is a constant.

Thus, it follows that *if one quantity varies inversely as another, their product is constant*, and conversely,

if the product of two quantities be constant, then either of them is said to vary inversely as the other.

Illustration : The number of days required to finish a piece of work varies inversely as the number of men employed. For a rectangle of given area, the length varies inversly as the breadth. If the demand of a commodity be constant, the price varies inversely as the supply.

5. Joint Variation.

One quantity is said to *vary jointly* as a number of others when it varies directly as their product.

Thus, A varies jointly as B, C,

if $A = m.BC$, where m is a constant.

Similarly, A varies jointly as B, C, D, E,

if $A = m, B. C. D. E$; etc.

One quantity is said to *vary directly* as a second and *inversely* as a third, when it *varies jointly* as the second and the reciprocal of the third.

Thus, A varies directly as B and inversely as C, if $A \propto \dfrac{B}{C}$, *i.e.*, if $A = m \cdot \dfrac{B}{C}$, where m is a constant.

Illustration : The quantity of work done varies jointly as the number of men employed and the number of days spent. The area of a triangle varies jointly as the base and altitude. The altitude of a triangle varies directly as the area and inversely as the base.

6. Elementary Properties.

It should be noted carefully that when two or more cases of variation occur in the same problem, *different constants of variation* should be used in different cases.

(i) *If $A \propto B$, then $B \propto A$.*

Since $A \propto B$, $\therefore$ $A = mB$, where m is a constant.

$\therefore$ $B = \dfrac{1}{m} A$; $\therefore$ $B \propto A$, since $\dfrac{1}{m}$ is constant.

(ii) *If $A \propto B$ and $B \propto C$, then $A \propto C$.*

Since $A \propto B$, $\therefore$ $A = mB$, where m is a constant.

Since $B \propto C$. $\therefore$ $B = nC$, where n is a constant.

$\therefore$ $A = mnC$, $\therefore$ $A \propto C$, for mn is a constant.

(iii) *If $A \propto C$ and $B \propto C$, then $(A \pm B) \propto C$ and $\sqrt{AB} \propto C$.*

Since $A \propto C$, $\therefore$ $A = mC$, where m is a constant.

Since $B \propto C$, $\therefore$ $B = nC$, where n is a constant.

$\therefore$ $(A \pm B) = (m \pm n)C$, $\therefore$ $(A \pm B) \propto C$.

Again, since $AB = mnC^2$, $\therefore$ $\sqrt{AB} = \sqrt{mn}.C$.

$\therefore$ $\sqrt{AB} \propto C$, since $\sqrt{mn}$ is a constant.

(iv) *If $A \propto B$, then $A^n \propto B^n$.*

Here $A = mB$. $\therefore$ $A^n = m^n B^n$. $\therefore$ $A^n \propto B^n$.

(v) *If* $A \propto B$ *and* $C \propto D$, *then* $AC \propto BD$ *and* $\dfrac{A}{C} \propto \dfrac{B}{D}$.

Here, $A = mB$ and $C = nD$, where m and n are constants.

$\therefore \quad AC = mnBD$, $\therefore AC \propto BD$, since mn is a constant.

Also $\dfrac{A}{C} = \dfrac{m}{n} \cdot \dfrac{B}{D}$, $\therefore \dfrac{A}{C} \propto \dfrac{B}{D}$, since $\dfrac{m}{n}$ is a constant.

(vi) *If* $A \propto BC$, *then* $B \propto \dfrac{A}{C}$ *and* $C \propto \dfrac{A}{B}$.

Here, $A = mBC$, $\therefore \quad B = \dfrac{1}{m} \cdot \dfrac{A}{C}$ and $C = \dfrac{1}{m} \cdot \dfrac{A}{B}$.

$\therefore \quad B \propto \dfrac{A}{C}$ and $C \propto \dfrac{A}{B}$, since $\dfrac{1}{m}$ is a constant.

7. Illustrative Examples.

Ex. 1. *If* $x+y \propto x-y$, *show that*

    (i) $x^2 + y^2 \propto xy$.

    (ii) $ax+by \propto px+qy$.   [C. U. 1936]

(i) Since, $x+y \propto x-y$, $\therefore x+y = m(x-y)$, m being a constant.

$\therefore$ Squaring, $x^2 + y^2 + 2xy = m^2(x^2 + y^2 - 2xy)$,

$\therefore \quad (m^2-1)(x^2+y^2) = 2(m^2+1)xy$,

$\therefore \quad x^2+y^2 = 2 \cdot \dfrac{m^2+1}{m^2-1} \cdot xy = k.xy$, say where $k = 2 \cdot \dfrac{m^2+1}{m^2-1}$.

$\therefore \quad x^2+y^2 \propto xy$, since k is a constant.

(ii) Since, $x+y \propto x-y$, $\qquad \therefore \dfrac{x+y}{x-y} = m$.

$\therefore$ By Comp. and Divd., $\dfrac{x}{y} = \dfrac{m+1}{m-1} = k$ suppose.

$\therefore x = yk$, (k is a constant, since m is a constant).

Now, $\dfrac{ax+by}{px+qy} = \dfrac{aky+by}{pky+qy} = \dfrac{ak+b}{pk+q} = $ constants.

$\therefore \qquad ax+by \propto px+qy$.

Ex. 2. *If $a \propto b$ and $b \propto c$, prove that*

$$a + b + c \propto \sqrt{bc} + \sqrt{ca} + \sqrt{ab}.$$

Here $b \propto c$, $\therefore b = mc$; and $a \propto b$, $\therefore a = nb = mnc$.

$$\therefore \frac{a + b + c}{\sqrt{bc} + \sqrt{ca} + \sqrt{ab}} = \frac{c(mn + m + 1)}{(\sqrt{m} + \sqrt{mn} + \sqrt{m^2 n})c}$$

$$= \frac{mn + m + 1}{\sqrt{m} + \sqrt{mn} + \sqrt{m^2 n}} = \text{const.}$$

$$\therefore a + b + c \propto \sqrt{bc} + \sqrt{ca} + \sqrt{ab}.$$

Ex. 3. *If x, y, z be variables such that $y + z - x$ is constant and $(z + x - y)(x + y - z) \propto yz$, prove that $x + y + z \propto yz$.*

Let $y + z - x = k$ and $(z + x - y)(x + y - z) = lyz$, where k and l are constants.

Now, $(x + y + z)(y + z - x) + (z + x - y)(x + y - z)$

$$= \{(y + z) + x\}\{(y + z) - x\} + \{x - (y - z)\}\{x + (y - z)\}$$

$$= \{(y + z)^2 - x^2\} + \{x^2 - (y - z)^2\}$$

$$= (y + z)^2 - (y - z)^2 = 4yz.$$

$$\therefore (x + y + z)k + lyz = 4yz.$$

$$\therefore (x + y + z) = \frac{4 - l}{l} \cdot yz.$$

$$\therefore x + y + z \propto yz, \text{ since } \frac{4 - l}{k} \text{ is constant.}$$

Ex. 4. *Two globes of gold have their radii equal to r and r'; they are melted and formed into a single globe. Find its radius. (The volume of a globe varies as the cube of the radius.)*   [*C. U. 1931*]

Let V and V' be the volumes of the globes. Then since the volume varies as the cube of the radius,

$$\left.\begin{array}{l} \therefore \quad V = kr^3 \\ \text{and } V' = kr'^3 \end{array}\right\} \text{ where } k \text{ is the constant of variation.}$$

Let R be the radius of the single globe, whose volume is $V + V'$.

$$\therefore \quad (V + V') = kR^3, \therefore k(r^3 + r'^3) = kR^3.$$

$$\therefore \quad R \text{ (radius of the single globe)} = \sqrt[3]{r^3 + r'^3}.$$

8. Theorem on Joint Variation.

If A varies as B when C is constant and A varies as C when B is constant, then A varies as BC when both B and C vary.

Suppose A has the value a_1 when B has the value b_1 and C has the value c_1.

Let A change from a_1 to a' when B changes from b_1 to b_2, and C remains constant at the value c_1.

$$\therefore \text{ by hypothesis, } \frac{a_1}{a'} = \frac{b_1}{b_2}. \qquad \cdots \qquad \cdots \quad (1)$$

Now, let A change from a' to a_2 when C changes from c_1 to c_2, and B remains constant at the value b_2.

$$\therefore \text{ by hypothesis, } \frac{a'}{a_2} = \frac{c_1}{c_2}. \qquad \cdots \qquad \cdots \quad (2)$$

Multiplying (1) and (2), we get

$$\frac{a_1}{a'} \times \frac{a'}{a_2} = \frac{b_1}{b_2} \times \frac{c_1}{c_2}, \text{ or } \frac{a_1}{a_2} = \frac{b_1 c_1}{b_2 c_2}.$$

Since a_1, b_1, c_1 and a_2, b_2, c_2 are two sets of simultaneous values of A, B, C and we find that when BC changes from $b_1 c_1$ to $b_2 c_2$, A changes from a_1 to a_2, in the same proportion, it follows, $A \propto BC$ when both B and C vary.

Alternative Proof.

Since $A \propto B$, when C is constant.

(1) $\therefore A = mB$, when m is constant with respect to A and B (*i.e.*, m is independent of the values of A and B).

Again, since $A \propto C$, when B is constant,

$$\therefore mB \propto C, \text{ when } B \text{ is constant,}$$

i.e., $m \propto C$, when B is constant.

(2) $\therefore m = nC$, when n is constant with respect to m and C, and since m is constant with respect to A and B n is constant with respect to A, B and C.

$\therefore$ from (1) and (2), we get $A = mB = nBC$.

Since n is constant with respect to A, B and C.

$$\therefore A \propto BC, \text{ when both } B \text{ and } C \text{ vary.}$$

Illustration : The area of a triangle varies as the base when the height remains constant, and the area varies as the height, when the base remains constant; hence when *both the base and the height vary, the area varies* **jointly** *as the base and the height.*

The principle of the above theorem will be clear from the following illustration from Simple Interest.

Interest (A)	Principal (B)	Time (C) .
Rs. 5 (a_1)	Rs. 100 (b_1)	1 year (c_1)
Rs. 25 (a')	Rs. 500 (b_2)	1 year (c_1)
Rs. 100 (a_2)	Rs. 500 (b_2)	4 years (c_2)

Cor. 1. *If $A \propto B$ when C and D are constants, $A \propto C$ when B and D are constants, $A \propto D$ when B and C are constants, then $A \propto BCD$ when B, C, D all vary.*

$A \propto B$ when C is constant and $A \propto C$ when B is constant. $\therefore$ by the preceding theorem, $A \propto BC$ when both B and C vary and by hypothesis D is constant in this case.

Thus, $A \propto BC$ when D is constant.

Also, $A \propto D$ when BC is constant.

$\therefore A \propto BCD$ when B, C, D all vary.

This is a generalisation of the preceding theorem and it can be still further generalised. Thus,

If A depend upon A_1, A_2,.........A_n, and on these alone, and vary as any one of these when the rest remain constant, then $A \propto A_1 A_2 ...... A_n$ when all vary.

Cor. 2. *If $A \propto B$ when C is constant and $A \propto \dfrac{1}{C}$ when B is constant, then $A \propto \dfrac{B}{C}$ when both B and C vary.*

When C is constant, $\dfrac{1}{C}$ is also constant and when C varies, $\dfrac{1}{C}$ also varies. Hence, by the application of the preceding theorem the required result follows.

Ex. *Apply the principle of variation to find how long 25 men will take to plough 30 acres, if 5 men take 9 days to plough 10 acres of land.*
[C. U. 1934]

Let $m=$ the number of men, $d=$ the number of days and $a=$ the number of acres. Then it is clear that m varies directly as a, when d is constant and inversely as d, when a is constant.

$\therefore$ $m \propto \dfrac{a}{d}$, when both a and d vary.

$\therefore$ $m = k \cdot \dfrac{a}{d}$, when k is constant of variation.

Now, when $m=5$, we have $a=10$, and $d=9$. $\therefore$ $5 = k \cdot \frac{10}{9}$.

$\therefore$ $k = \dfrac{9}{2}$. $\therefore$ $m = \dfrac{9}{2} \cdot \dfrac{a}{d}$.

$\therefore$ when $m=25$, and $a=30$, we have

$25 = \dfrac{9}{2} \cdot \dfrac{30}{d}$. $\therefore$ $d = \dfrac{9 \times 30}{2 \times 25} = \dfrac{27}{5}$.

Examples II

1. If x varies directly as y and inversely as z and if $x=a$, when $y=b$ and $z=c$, find the value of x, when $y=b^2$ and $z=c^2$.

2. If A varies as B and C jointly and if $A=2$, when $B=\frac{3}{5}$, $C=\frac{10}{7}$, find C when $A=54$ and $B=\frac{3}{5}$.

3. If $x^2 \propto yz$, $y^2 \propto zx$ and $z^2 \propto xy$, find the relation between the three constants of variation.

4. (i) If $\dfrac{x}{y} \propto x+y$ and $\dfrac{y}{x} \propto x-y$, show that

$(x^2 - y^2)$ is invariable.

(ii) If $x \propto 1/y$, prove that $x+y$ is least when $x=y$.

(iii) If $x^3 + \dfrac{1}{y^3} \propto x^3 - \dfrac{1}{y^3}$, then $y \propto \dfrac{1}{x}$.

5. (i) If $x = a + \beta + \gamma$, when a is a constant, $\beta \propto y$ and $\gamma \propto y^2$ and if $x = 6, 17, 34$ when $y = 1, 2, 3$ respectively ; find the equation connecting x and y.

(ii) If y varies as the sum of two quantities of which one varies directly as x and the other inversely as x ; and if $y = 6$ when $x = 4$ and $y = 15$ when $x = 8$; find the relation between x and y.

6. If $x \propto y$ and $y \propto z$ and if a, b, c and a', b', c' be two sets of values of x, y, z, show that

$$\frac{a^2 + b^2 + c^2}{aa' + bb' + cc'} = \frac{aa' + bb' + cc'}{a'^2 + b'^2 + c'^2}.$$

7. (i) If $x \propto y$ and $y \propto z$, show that

$$x^5 + x^2 z^3 + yz^4 \propto x^5 + y^5 + z^5.$$

(ii) If y is equal to the sum of two quantities of which one varies directly as x and the other inversely as x^2, and if $y = 49$, when $x = 3$, or 5, express y in terms of x.

8. (i) If $x \propto 1/y$, when z is constant $x \propto 1/z$, when y is constant, prove that $x \propto 1/yz$, when both y and z are variable.

(ii) If $x \propto y$, when z is constant and $x \propto z$, when y is constant, prove that when $y \propto z$, then will $x \propto y^2$, or z^2.

(iii) If $x \propto y + z$, when $y - z$ is constant and $x \propto y - z$, when $y + z$ is constant, then will $x \propto y^2 - z^2$, when both y and z vary.

9. Given that $x + y \propto z + \dfrac{1}{z}$ and $x - y \propto z - \dfrac{1}{z}$, and the relation between x and z, if $z = 2$, when $x = 3$ and $y = 1$.

10. If $x \propto \dfrac{z}{y^2}$ and $z^2 \propto \dfrac{y}{x}$, show that $x \propto \dfrac{1}{y} \propto \dfrac{1}{z}$.

11. (i) If $ax + by \propto cx + dy$, then $x \propto y$.

(ii) If $ax^2 + by^2 \propto cx^2 + dy^2$, then $x \propto y$.

12. If $x^2 + y^2 \propto x^2 - y^2$, then

   (i) $x + y \propto x - y$.   (ii) $x \propto y$.

13. If $x + y \propto x - y$, then

   (i) $x^3 + y^3 \propto x^3 - y^3$.   (ii) $x^3 + y^3 \propto xy(x \pm y)$.

14. If $x + y \propto z$ and $y + z \propto x$, prove that $z + x \propto y$.

15. If $x \propto y$ and $y \propto z$, prove that

   (i) $x^2 + y^2 + z^2 \propto yz + zx + xy$.

   (ii) $x^3 + y^3 + z^3 \propto 3xyz$.

   (iii) $x^n + y^n + z^n \propto yz^{n-1} + zx^{n-1} + xy^{n-1}$.

   (iv) $ax + by + cz \propto l\sqrt{yz} + m\sqrt{zx} + n\sqrt{xy}$.

16. If $x + y \propto z$, when y is constant, and $x + z \propto y$, when x is constant, show that when both y and z vary, then $x + y + z \propto yz$.   [*C. U. 1941*]

17. The volume of a sphere varies as the cube of the radius and the surface of a sphere varies as the square of the radius. Show that the square of the volume varies as the cube of the surface.   [*C. U. 1924*]

18. It is given that the illumination from a source of light varies inversely as the square of the distance ; how much further from a table-lamp must a book, which is now 6 inches off, be removed so as to receive just half as much light ?

19. Three globes of gold whose radii are 3, 4 and 5 inches are melted and formed into a new globe ; find the radius of the new globe, given that the volume of a globe varies as the cube of its radius.

20. The distance through which a heavy body fells from rest varies as the square of the time it falls ; also a body falls 64 feet in 2 seconds. How far does a body fall in 6 seconds ?

2

21. The time of oscillation of a pendulum varies as the square root of its length. If the pendulum of length 40 inches oscillate once in a second, what is the length of the pendulum oscillating once in 2˙5 sec. ?

22. Apply the principle of variation to find how long will 15 men to mow 80 bighas of grass, if 10 men take 18 days to mow 60 bighas.

23. The expenses of a boarding house are partly constant and partly vary as the number of boarders. When the numbers of boarders are 450 and 920, the expenses are Rs. 1485 and Rs. 2895 respectively. Find the expenses for 1000 boarders.

24. The pressure in a liquid varies as depth, when the density is constant and it varies as density, when the depth is constant. The pressure is 1, when the depth is 32 and density is 1. Find the depth at which the pressure is 2, when the density is 16.   [*C. U. 1921*]

25. The value of a silver coin varies as the square of its diameter, when its thickness remains the same and varies as its thickness, when its diameter remains the same. Two silver coins have their diameters in the ratio of 4 : 3 ; find the ratio of their thickness, if the value of the first be four times that of the second.

26. The volume of a given mass of gas varies directly as the absolute temperature, when the pressure is constant and inversely as the pressure, when the temperature is constant. If the volume be 50 cubic inches, when the pressure is 60 inches of mercury and the absolute temperature 300, find the volume, when the pressure is 70 inches of mercury and the absolute temperature 280.

27. The attendance at a Professor's lecture varies directly as the Professor's power of exposition and inversely as the square of the number of lectures delivered. If 64 students attend the lecture of Professor X who delivers a course of 12 lectures, find the number of students who

attend the lectures of Professor Y who delivers a course of 16 lectures and who possesses twice as much power of exposition as Professor X.

28. If an income-tax varies as the square of the excess of a man's income over Rs. 3000, and if this tax on an income of Rs. 6000, amounts to Rs. 100, show that under this system a man whose income is Rs. 96,000 would be made a pauper.

29. A locomotive engine without carriages can go 40 miles an hour, and its speed is diminished by a quantity which varies as the square root of the number of carriages attached ; with 16 carriages its speed is 20 miles an hour. Find the least number of carriages which the engine fails to move.

30. Supposing the weight of an empty ship and the tonnage to vary as the square and the cube of its length respectively, prove that (w_1 being the weight with cargo of a ship of length l_1 and w_2, w_3 being similar weights corresponding to lengths l_2 and l_3)

$$\frac{w_1}{l_1^2}(l_2 - l_3) + \frac{w_2}{l_2^2}(l_3 - l_1) + \frac{w_3}{l_3^2}(l_1 - l_2) = 0.$$

31. At a certain foot-ball competition, the number of matches on each day varies jointly as the number of days from the beginning and end of the competition up to and including the day in question. On three successive days there were respectively 6, 5 and 3 matches. Which days were these and how long did the competition last ?

32. The consumption of coal by a locomotive engine varies as the square of the velocity. When the velocity is 50 miles per hour, the consumption of coal is 100 maunds per hour. If the price of coal be 4as. per maund and the other expenses of the engine be Rs. 9 per hour, find the least cost of a journey of 250 miles.

ANSWERS

1. $\dfrac{ab}{c}$.   2. $38\frac{4}{7}$.

3. If l, m, n be the constants, $lmn = 1$.

5. (i) $x = 1 + 2y + 3y^2$.   (ii) $y = 2x - 8x^{-1}$.

7. (ii) $y = 8x + 225x^{-2}$.   9. $15zx = 2(11z^2 + 1)$.

18. $6(\sqrt{2} - 1)$.   19. 6 inches.   20. 576 ft.

21. 250 inches.   22. 16 days.   23. Rs. 3135.

24. 4.   25. $9 : 4$.   26. 40 cubic inches.

27. 72.   29. 64.

31. 4th, 5th and 6th days ; 6 days.   32. Rs. 150.

9. Involution and Evolution.

The process of determining powers of quantities is often called *involution* and the inverse process, namely that by which roots of the quantities are obtained is called *evolution*.

10. Fundamental Laws of Indices.

If m and n are positive integers,

(i) $a^m \times a^n = a^{m+n}$.

(ii) $a^m \div a^n = a^{m-n}$, ($m > n$).

(iii) $(a^m)^n = a^{mn} = (a^n)^m$.

(iv) $(ab)^n = a^n b^n$.
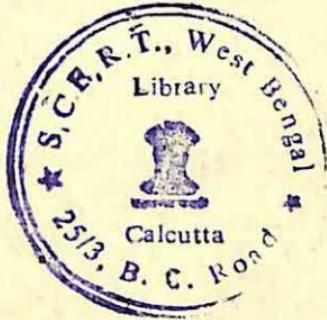
The students are supposed to be already familiar with the proofs of these elementary laws of indices, when m and n are positive integers.

11. Fractional, Zero and Negative Indices.

We do not know the meaning of a^m when m is not a positive integer. Whatever meaning may be assigned to the symbol a^m, when m is not a positive integer, in order that this meaning may be useful it must be such that the fundamental laws of indices continue to hold good.

We interpret a^m when m is (i) fractional, (ii) zero and (iii) negative by *assuming that the fundamental index law* $a^m \times a^n = a^{m+n}$ is true for all values of m and n (positive, negative, zero, integral or fractional) and then show that with these new meanings of the symbol a^m, the other laws of indices are also true for all values[*] of m and n.

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[*] Here only rational values of indices are considered.
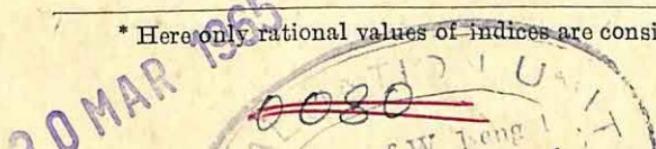
(i) Interpretation of $a^{\frac{p}{q}}$, *p and q being positive integers*.

Since the index-law $a^m \times a^n = a^{m+n}$ has been assumed true for all values of m and n, therefore, putting $m = n = \dfrac{p}{q}$, we have

$$a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{p}{q}+\frac{p}{q}} = a^{\frac{2p}{q}}.$$

Similarly, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times a^{\frac{p}{q}} = a^{\frac{2p}{q}} \times a^{\frac{p}{q}} = a^{\frac{3p}{q}}$; and so on.

Hence, $a^{\frac{p}{q}} \times a^{\frac{p}{q}} \times \cdots$ to q factors $= a^{q \cdot \frac{p}{q}} = a^p$.

$$\therefore \quad \left(a^{\frac{p}{q}} \right)^q = a^p.$$

Thus, since the qth power of $a^{\frac{p}{q}}$ is a^p,

$\therefore$ $a^{\frac{p}{q}}$ *represents the qth root of the pth power of a, so*

$$a^{\frac{p}{q}} = \sqrt[q]{a^p}.$$

Cor. 1. Putting $p = 1$, we see that $a^{\frac{1}{q}}$ *represents the qth root of a i.e.,* $a^{\frac{1}{q}} = \sqrt[q]{a}$.

Cor. 2. Again, as before, we have

$$a^{\frac{1}{q}} \times a^q \ldots\ldots\ldots\ldots\ldots \text{ to } p \text{ factors}$$

$$= a^{\frac{1}{q}+\frac{1}{q}+\cdots \text{ to } p \text{ terms}} = a^{\frac{p}{q}}.$$

$$\therefore \quad \left(a^{\frac{1}{q}} \right)^p = a^{\frac{p}{q}}, \qquad \therefore \quad a^{\frac{p}{q}} = (\sqrt[q]{a})^p.$$

Thus, $a^{\frac{p}{q}}$ *can also be regarded as the pth power of the qth root of a.*

Note. It should be noted that it is not strictly true that $\sqrt[q]{a^p} = (\sqrt[q]{a})^p$, unless by the qth root of a quantity is meant only the arithmetical root or except with certain other limitations. For example,

$$\sqrt[2]{a^4} = \pm a^2, \text{ whereas } (\sqrt[2]{a})^4 = +a^2 \text{ only.}$$

(ii) Interpretation of a^0.

Since $a^m \times a^n = a^{m+n}$ has been assumed as true for all values of m and n, therefore, putting $m = 0$, we get

$$a^0 \times a^n = a^{0+n} = a^n.$$

Dividing both sides by a^n (assuming $a \neq 0$), we get

$$a^0 = a^n/a^n = 1.$$

$\therefore$ *for all values of a except zero, a^0 is equivalent to* 1.

(iii) Interpretation of a^{-n}, n being a positive integer.

Since the index law $a^m \times a^n = a^{m+n}$ has been assumed to be true for all values of m and n, therefore

$$a^m \times a^n \times a^{-n} = a^{m+n} \times a^{-n} = a^{m+n-n} = a^m.$$

Dividing both sides by a^m (assuming $a \neq 0$), we get

$$a^n \times a^{-n} = 1.$$

$$\therefore \quad a^{-n} = \frac{1}{a^n}.$$

Thus, a^{-n} *means the reciprocal of* a^n.

12. Generalised Laws of indices.

To prove that *for all values of m and n,*

(A) $a^m \div a^n = a^{m-n}$.

(B) $(a^m)^n = a^{mn} = (a^n)^m$.

(C) $(ab)^n = a^n b^n$.

(A) Since $a^m = a^n \times a^{m-n}$ is true for all values of m and n, we have $a^m \div a^n = a^{m-n}$ for all values of m and n.

(B) *Case I. Let n be a positive integer.*

Then, whatever m may be

$$(a^m)^n = a^m \times a^m \times \cdots\cdots \text{ to } n \text{ factors}$$
$$= a^{m+m+\cdots \text{ to } n \text{ terms}} = a^{mn}.$$

Case II. Let n be a positive fraction $= p/q$, p, q being positive integers, then whatever m may be,

$$(a^m)^n = (a^m)^{p/q} = \sqrt[q]{(a^m)^p} = \sqrt[q]{a^{mp}} = a^{mp/q} = a^{mn}.$$

Case III. Let n be negative and $= -p$, where p is a positive quantity, fractional or integral. Now whatever m may be,

$$(a^m)^n = (a^m)^{-p} = \frac{1}{(a^m)^p} = \frac{1}{a^{mp}} = a^{-mp} = a^{mn}.$$

$\therefore$ $(a^m)^n = a^{mn}$ for all values of m and n.

(C) *Case I. Let n be a positive fraction $= p/q$, where* p and q are positive integers. Then

$$(ab)^n = (ab)^{p/q} = \sqrt[q]{(ab)^p} = \sqrt[q]{a^p b^p} = \sqrt[q]{a^{nq} b^{nq}} = \sqrt[q]{(a^n b^n)^q}$$
$$= (a^n b^n)^{q/q} = a^n b^n.$$

Case II. Let n be a negative quantity, say $(-p)$, then p is a positive quantity, integral or fractional.

$$(ab)^n = (ab)^{-p} = \frac{1}{(ab)^p} = \frac{1}{a^p b^p} = a^{-p} b^{-p} = a^n b^n.$$

$\therefore$ $(ab)^n = a^n b^n$ for all values of n.

Cor. 1. $(abcd\cdots)^n = a^n b^n c^n d^n \cdots\cdots$

Cor. 2. $\left(\dfrac{a}{b}\right)^n = \left(a \times \dfrac{1}{b}\right)^n = (a \cdot b^{-1})^n = a^n (b^{-1})^n = a^n b^{-n} = \dfrac{a^n}{b^n}.$

Cor. 3. $(a^x b^x c^x \cdots\cdots)^n = a^{xn} b^{xn} c^{xn} \cdots\cdots$

13. Illustrative Examples.

Ex. 1. *Show that*

$$\left(\frac{x^b}{x^c}\right)^{b+c} \times \left(\frac{x^c}{x^a}\right)^{c+a} \times \left(\frac{x^a}{x^b}\right)^{a+b} = 1.$$

Left side $= (x^{b-c})^{b+c} \times (x^{c-a})^{c+a} \times (x^{a-b})^{a+b}$

$$= x^{b^2-c^2} \times x^{c^2-a^2} \times x^{a^2-b^2}$$

$$= x^{b^2-c^2+c^2-a^2+a^2-b^2} = x^0 = 1.$$

Ex. 2. *If* $a=b^x$, $b=c^y$ *and* $c=a^z$, *then* $xyz=1$.

$\because\ b=c^y.$ $\qquad\qquad \therefore\ b^x=c^{xy}.$

$\because\ c=a^z.$ $\qquad\qquad \therefore\ c^{xy}=a^{xyz}.$

$\because\ a=b^x=c^{xy}=a^{xyz}.$ $\quad \therefore\ xyz=1.$

Ex. 3. *Solve* $27^x=9^y$; $81^y=243\times3^x$.

From 1st equation, $(3^3)^x=(3^2)^y$,

or, $3^{3x}=3^{2y}.$ $\qquad\qquad \therefore\ 3x=2y.$ $\qquad\qquad$ ... (1)

From 2nd equation, $(3^4)^y=3^5\times3^x$.

$\therefore\ 3^{4y}=3^{5+x}.$ $\qquad\qquad \therefore\ 5+x=4y.$ $\qquad\qquad$ ... (2)

$\therefore\ 5+x=2.2y=2.3x$, from (1), $=6x$.

$\therefore\ x=1$, and from (1), $y=\frac{3}{2}=1\frac{1}{2}.$

Note. Equations of the above type in which the unknown quantities appear as exponents are called *Exponential equations.*

Examples III

Show that (Exs. 1-5) :

1. $\sqrt{x^{-1}}.y \times \sqrt{y^{-1}}.z \times \sqrt{z^{-1}}.x = \sqrt{xyz}.$

2. $\left(\frac{x^b}{x^c}\right)^a \times \left(\frac{x^c}{x^a}\right)^b \times \left(\frac{x^a}{x^b}\right)^c = 1.$

3. $\left(a^{\frac{1}{x-y}}\right)^{\frac{1}{x-z}} \times \left(a^{\frac{1}{y-z}}\right)^{\frac{1}{y-x}} \times \left(a^{\frac{1}{z-x}}\right)^{\frac{1}{z-y}} = 1.$

4. $\left(\dfrac{x^b}{x^c}\right)^{\frac{1}{bc}} \times \left(\dfrac{x^c}{x^a}\right)^{\frac{1}{ca}} \times \left(\dfrac{x^a}{x^b}\right)^{\frac{1}{ab}} = 1.$

5. $\dfrac{1}{1+x^{b-a}+x^{c-a}} + \dfrac{1}{x^{a-b}+1+x^{c-b}} + \dfrac{1}{x^{a-c}+x^{b-c}+1} = 1.$

6. If $a = xy^{p-1}$, $b = xy^{q-1}$, $c = xy^{r-1}$, show that
$$a^{q-r}b^{r-p}c^{p-q} = 1.$$

7. If $a = x^{q+r}y^p$, $b = x^{r+p}y^q$, $c = x^{p+q}y^r$, show that
$$a^{q-r}b^{r-p}c^{p-q} = 1.$$

8. Solve :

    (i) $2^{x+3} + 2^{x+1} = 320.$ (ii) $ba^{x-2} = ab^{x-2}.$

    (iii) $a^x + b^y = a + b,$ (iv) $x^y = y^x$, $x = 2y.$
        $a^{x-1} + b^{y-1} = 2.$ [C. U. 1935]

    (v) $3^x - 2^y = 1,$ (vi) $x^y = y^x$, $x^2 = y^3.$
        $3^{x+1} + 2^{y-1} = 31.$ (vii) $a^x = x^y$, $a^y = x^x.$

    (viii) $a^x = (x+y+z)^y$, $a^y = (x+y+z)^z$, $a^z = (x+y+z)^x.$

9. If $a^x = b^y = c^z$ and $b^2 = ac$, prove that

$$\frac{1}{x} + \frac{1}{z} = \frac{2}{y}.$$
                                   [C. U. 1944]

10. If $m = a^x$, $n = a^y$, $a^2 = (m^y n^x)^z$, show that $xyz = 1.$

11. If $x^y = y^x$, show that $\left(\dfrac{x}{y}\right)^{\frac{x}{y}} = x^{\frac{x}{y}-1}$, and if $x = 2y$,
prove that $y = 2.$ [C. U. 1928]

12. If $a = (\sqrt{2}-1)^{\frac{1}{2}}$ show that
$$(a - a^{-1})^3 + 3(a - a^{-1}) + 2 = 0.$$

13. If $x = \sqrt[3]{\{a + \sqrt{a^2 + b^3}\}} + \sqrt[3]{\{a - \sqrt{a^2 + b^3}\}}$, show that
$$x^3 + 3bx - 2a = 0.$$

14. If $a\sqrt[3]{x^2} + b\sqrt[3]{x} + c = 0$, show that
$$a^3x^3 + b^3x + c^3 = 3abcx.$$

15. If $\sqrt{\{(x - \sqrt{a^2 - b^2})^2 + y^2\}} + \sqrt{\{(x + \sqrt{a^2 - b^2})^2 + y^2\}}$
$= 2a$, show that $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

16. Define $a^{\frac{p}{q}}$ as $\sqrt[q]{a^p}$ and a^{-n} as $\dfrac{1}{a^n}$; hence prove that
the laws $a^m.a^n = a^{m+n}$, $(a^m)^n = a^{mn}$, $(ab)^m = a^m b^m$ hold when
$m = -\frac{1}{4}$, $n = -\frac{1}{5}$.

ANSWERS

8. (i) $x = 5$.   (ii) $x = 3$.   (iii) $x = y = 1$.

 (iv) $x = 4$, $y = 2$.   (v) $x = 2$, $y = 3$.   (vi) $x = \frac{27}{8}$, $y = \frac{9}{4}$.

 (vii) $x = y = a$, or, $x = a^{-1}$, $y = -a^{-1}$.   (viii) $x = y = z = \frac{1}{3}a$.

14. Definition and classification of Surds.

When a root of a quantity cannot be exactly determined the root is called a *surd*. Thus, $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[3]{5}$ etc. are surds.

Quantities which can be expressed as the ratio of two integers are called *Rational* quantities. Thus, 5, $\frac{3}{7}$, $\sqrt{4}$ are rational quantities.

Quantities which cannot be so expressed are called *Irrational quantities*; thus, $\sqrt{7}$, $\sqrt[3]{3\pi}$ etc. are irrational quantities.

It is also clear *that all surds are incommensurable*. Thus, $\sqrt{2}$ cannot be exactly expressed as the ratio of two integers, although its value can be obtained to any desired degree of accuracy.

It should also be noted that an algebraic expression, such as $\sqrt{a}$, is often called a *surd*, although a may have such a value that $\sqrt{a}$, is not in reality a surd. Thus, if $a = 9$, $\sqrt{a} = \sqrt{9} = 3$ and is therefore not really a surd.

The *order* of a surd is denoted by the index of the root to be extracted.

Thus, $\sqrt{2}$, $\sqrt[4]{5}$, $\sqrt[n]{a}$ are surds of the 2nd, 4th and nth orders respectively.

Surds of the second order, are called *quadratic surds*, those of the third order, *cubic surds*, those of the fourth order, *biquadratic surds*, and so on.

Surds are said to be of the *same order*, or *of different orders* according as their surd indices are the same or different.

Thus, $\sqrt{a}$, $a^{\frac{3}{6}}$ are surds of the same order, but $\sqrt[3]{a}$, $\sqrt[5]{a}$ are of different orders.

Surds which consist of a rational and a surd factor are called *Mixed* surds, while surds which have no rational factors are called *Pure* surds.

Thus, $2\sqrt{5}$ is a mixed surd and $\sqrt{7}$ is a pure surd.

The algebraic sum of two surds or a surd and a rational quantity is called a *binomial surd*.

Thus, $3\sqrt{2} + 4\sqrt{3}$, $3 + \sqrt{5}$, $\sqrt[3]{25} - \sqrt[3]{5}$ are called binomial surds.

Similarly, $\sqrt{3} + \sqrt{7} + 5$, $1 + \sqrt{2} + \sqrt{3}$ are trinomial surds.

When a surd consists of a single term it is called a *simple surd*, but the algebraic sum of two or more surds is called a *compound surd*.

Thus, $\sqrt{5}$ is a simple surd, but $2\sqrt{7} + \sqrt{3} + \sqrt{2}$ is a compound surd.

Surds are said to be *like* or similar when they have or can be so reduced as to have the same surd-factor.

Surds which are not like or similar are said to be *unlike* or *dissimilar*.

Thus, $\sqrt{20}$ and $\sqrt{45}$ are similar surds, since, they can respectively be written as $2\sqrt{5}$ and $3\sqrt{5}$. But $\sqrt{3}$ and $\sqrt{5}$ are dissimilar surds ; $\sqrt{2}$ and $\sqrt[4]{2}$ are also dissimilar.

15. Rationalisation of Surds.

If two surds be such that their product is rational, each is called a *rationalising factor* of the other and *each is said to be rationalised by the other.*

Thus, since $(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = 5 - 3 = 2$, it is clear that $\sqrt{5} + \sqrt{3}$ is made rational by multiplying by $\sqrt{5} - \sqrt{3}$. Hence, $\sqrt{5} - \sqrt{3}$ is a rationalising factor of $\sqrt{5} + \sqrt{3}$.

Similarly, since $(x + \sqrt{y})(x - \sqrt{y}) = x^2 - y$, $x \pm \sqrt{y}$ is made rational by multiplying by $x \pm \sqrt{y}$. So also $a\sqrt{b} \pm c\sqrt{d}$ is made rational by multiplying by $a\sqrt{b} \mp c\sqrt{d}$.

16.   Rationalising factor of Binomial Surd.

Case 1.   Let the binomial surd be $\sqrt[p]{a} - \sqrt[q]{b}$ or $a^{\frac{1}{p}} - b^{\frac{1}{q}}$.

Let $a^{\frac{1}{p}} = x$, $b^{\frac{1}{q}} = y$ and n be the L. C. M. of p and q, then x^n and y^n and both rational and as such $x^n - y^n$ is rational.

Now $x^n - y^n$ is divisible by $x - y$ whether n be an odd or even positive integer and $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots\cdots + y^{n-1})$.

Thus, the rationalising factor is
$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots\cdots + y^{n-1}$$
and the rational product is $x^n - y^n$.

Case 2.   Let the binomial surd be $\sqrt[p]{a} + \sqrt[q]{b}$ or $a^{\frac{1}{p}} + b^{\frac{1}{q}}$.

Let $a^{\frac{1}{p}} = x$, $b^{\frac{1}{q}} = y$ and n be the L. C. M. of p and q.

(a) *If n is even*, $x^n - y^n$ is divisible by $x + y$ and
$$x^n - y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots\cdots + xy^{n-2} - y^{n-1}).$$

Thus, the rationalising factor is $x^{n-1} - x^{n-2}y + \cdots\cdots + xy^{n-2} - y^{n-1}$ and the rational product is $x^n - y^n$.

(b) If *n is odd*, $x^n + y^n$ is divisible by $x + y$ and
$$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots\cdots - xy^{n-2} + y^{n-1}).$$

Thus, the rationalising factor is $x^{n-1} - x^{n-2}y + \cdots\cdots - xy^{n-2} + y^{n-1}$ and the rational product is $x^n + y^n$.

Ex. 1.   *Find the factor which will rationalise* $\sqrt{3} + \sqrt[3]{2}$.

Now $\sqrt{3} + \sqrt[3]{2} = 3^{\frac{1}{2}} + 2^{\frac{1}{3}}$.   Hence 6 is the L. C. M. of 2 and 3.

Let $x = 3^{\frac{1}{2}}$ and $y = 2^{\frac{1}{3}}$, then x^6, y^6, as also $x^6 - y^6$ are rational being respectively equal to 27, 4 and 23.

Now $x^6 - y^6 = (x + y)(x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5)$.

Hence, the rationalising factor of $x + y$ i,e., of $\sqrt{3} + \sqrt[3]{2}$ is
$$x^5 - x^4y + x^3y^2 - x^2y^3 + xy^4 - y^5,$$

i.e., $3^{\frac{5}{2}} - 9.2^{\frac{1}{3}} + 3^{\frac{3}{2}}2^{\frac{2}{3}} - 6 + 3^{\frac{1}{2}}2^{\frac{4}{3}} - 2^{\frac{5}{3}}$

i.e., $9\sqrt{3} - 9.\sqrt[3]{2} + 3\sqrt{3}.\sqrt[3]{4} - 6 + 2\sqrt{3}.\sqrt[3]{2} - 2\sqrt[3]{4}$.

Ex. 2. *Express with rational denominator*

$$\frac{\sqrt[3]{3}-1}{\sqrt[3]{3}+1}.$$

The denominator is $3^{\frac{1}{3}}+1$. Here the L. C. M. of the denominator of the indices of $3^{\frac{1}{3}}$ and 1 is 3. Now, $x^3+y^3=(x+y)(x^2-xy+y^2)$.

Here the rationalising factor of the denominator is

$$\left(3^{\frac{1}{3}}\right)^2-3^{\frac{1}{3}}.1+1, \text{ i.e., } 3^{\frac{2}{3}}-3^{\frac{1}{3}}+1.$$

Hence multiplying numerator and denominator of the given expression by $3^{\frac{2}{3}}-3^{\frac{1}{3}}+1$,

we have the given expression $= \dfrac{\left(3^{\frac{1}{3}}-1\right)\left(3^{\frac{2}{3}}-3^{\frac{1}{3}}+1\right)}{\left(3^{\frac{1}{3}}+1\right)\left(3^{\frac{2}{3}}-3^{\frac{1}{3}}+1\right)}$

$$= \frac{3-2.3^{\frac{2}{3}}+2.3^{\frac{1}{3}}-1}{3+1}$$

$$= \tfrac{1}{2}(1-\sqrt[3]{9}+\sqrt[3]{3}).$$

17. Conjugate (or Complementary) Surds.

Two binomial quadratic surds which differ only in the sign connecting their forms, are said to be *conjugate* or *complementary* to each other.

Thus, $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ or $a\sqrt{b}+c\sqrt{d}$ and $a\sqrt{b}-c\sqrt{d}$ are conjugate surds.

Since $(a\sqrt{b}+c\sqrt{d})(a\sqrt{b}-c\sqrt{d})=a^2b-c^2d$, which is rational, we conclude that the *rationalising factor of a binomial quadratic surd is its conjugate.*

18. Properties of Binomial Surds.

(i) *A quadratic surd cannot be equal to the sum or difference of a rational quantity and a quadratic surd.*

For, if possible, let $\sqrt{a}=b\pm\sqrt{c}$.

Squaring, we have $a=b^2+c\pm2b\sqrt{c}$.

$$\therefore \quad \sqrt{c}=\pm\frac{a-b^2-c}{2b}.$$

Thus, a surd is equal to a rational quantity, which is absurd. Hence our assumption is wrong and the theorem is true.

(ii) If $a + \sqrt{b} = c + \sqrt{d}$, *where a and c are rational and* $\sqrt{b}$ *and* $\sqrt{d}$ *are irrational*, then $a = c$ and $b = d$.

If a be not equal to c, let $a = c + m$,

$$\text{then } c + \sqrt{d} = a + \sqrt{b} = c + m + \sqrt{b} \ ;$$
$$\therefore \quad \sqrt{d} = m + \sqrt{b}.$$

$\therefore$ Squaring, $d = m^2 + b + 2m\sqrt{b}$;

$$\therefore \quad \sqrt{b} = \frac{d - m^2 - b}{2m}.$$

Thus, a surd is equal to a rational quantity, which is absurd. $\therefore$ $a = c$ and consequently $b = d$.

Cor. Similarly, it can be shown that if

$$a - \sqrt{b} = c - \sqrt{d}, \text{ then } a = c \text{ and } b = d.$$

Note. It is clear therefore that an equation like $a - \sqrt{b} = c + \sqrt{d}$ is equivalent to two independent equations *viz.*, $a = c$ and $b = d$.

If $a + \sqrt{b} = 0$, we must have separately $a = 0$, $b = 0$.

(iii) *If* $\sqrt{(a + \sqrt{b})} = \sqrt{c} + \sqrt{d}$, then

$$\sqrt{(a - \sqrt{b})} = \sqrt{c} - \sqrt{d}.$$

By squaring, $\sqrt{(a + \sqrt{b})} = \sqrt{c} + \sqrt{d}$, we have

$$a + \sqrt{b} = c + d + 2\sqrt{(cd)}.$$
$$\therefore \quad a = c + d \ ; \quad \sqrt{b} = 2\sqrt{(cd)}, \quad [\text{ by } (ii)\ ]$$
$$\therefore \quad a - \sqrt{b} = c + d - 2\sqrt{(cd)} = (\sqrt{c} - \sqrt{d})^2.$$
$$\therefore \quad \sqrt{(a - \sqrt{b})} = \sqrt{c} - \sqrt{d}.$$

Cor. If $\sqrt{(a - \sqrt{b})} = \sqrt{c} - \sqrt{d}$, then $\sqrt{(a + \sqrt{b})} = \sqrt{c} + \sqrt{d}$.

(iv) If $\sqrt[3]{(a + \sqrt{b})} = x + \sqrt{y}$, then $\sqrt[3]{(a - \sqrt{b})} = x - \sqrt{y}$.

By cubing, $\sqrt[3]{(a + \sqrt{b})} = x + \sqrt{y}$, we have

$$a + \sqrt{b} = x^3 + 3x^2 \sqrt{y} + 3xy + y \sqrt{y}$$
$$= (x^3 + 3xy) + (3x^2 + y) \sqrt{y}.$$

$\therefore$ Equating rational and irrational parts,

$$a = x^3 + 3xy \qquad \cdots \qquad \cdots \qquad (1)$$

and $\quad \sqrt{b} = (3x^2 + y) \sqrt{y} \cdots \qquad \cdots \qquad (2)$

Hence subtracting (2) from (1), we get

$$a - \sqrt{b} = x^3 - 3x^2 \sqrt{y} + 3xy - y \sqrt{y} = (x - \sqrt{y})^3.$$

$\therefore \quad \sqrt[3]{(a - \sqrt{b})} = x - \sqrt{y}.$

Cor. If $\sqrt[3]{(a - \sqrt{b})} = x - \sqrt{y}$, then $\sqrt[3]{(a + \sqrt{b})} = x + \sqrt{y}$.

Note 1. The above property will be found useful in extracting the cube root of $a + \sqrt{b}$. The method of obtaining the cube root of a binomial surd is illustrated in the example 9 of Art. 20.

Note 2. Generally.

If $\sqrt[n]{(a + \sqrt{b})} = x + \sqrt{y}$, then $\sqrt[n]{(a - \sqrt{b})} = x - \sqrt{y}$, n being a positive integer.

This can be easily established by the help of Binomial Theorem.

19. Square root of a Quadratic Surd.

The square of the sum of two quadratic surds

$\qquad\qquad$ = a rational quantity + a surd ;

thus $(\sqrt{3} + \sqrt{2})^2 = 3 + 2 + 2 \sqrt{6} = 5 + 2 \sqrt{6}.$

Similarly, $(\sqrt{a} + \sqrt{b})^2 = a + b + 2 \sqrt{ab} = (a + b) + \sqrt{4ab}.$

Thus, the square of a binomial quadratic surd of the form $\sqrt{x} + \sqrt{y}$, may be expressed in the form $a + \sqrt{b}$. Hence, the square root of a binomial quadratic surd of the form $a + \sqrt{b}$ is of the form $\sqrt{x} + \sqrt{y}$.

To find the square root of $a + \sqrt{b}$.

Let $\sqrt{(a + \sqrt{b})} = \sqrt{x} + \sqrt{y}$.

Squaring, $a + \sqrt{b} = x + y + 2\sqrt{xy}$.

$\therefore$ By Art. 18(ii), we have

$$x + y = a \qquad \cdots \qquad \cdots \qquad (i)$$

and $\quad 2\sqrt{xy} = \sqrt{b}$ i.e., $4xy = b.$ $\cdots$ $\qquad \cdots$ $\quad$ (ii)

$\therefore (x + y)^2 - 4xy = a^2 - b.$

$\therefore (x - y)^2 = a^2 - b.$ $\quad \therefore x - y = \sqrt{(a^2 + b)}.$

Since, $\quad x + y = a \qquad \cdots \qquad \cdots \qquad (iii)$

and $\quad x - y = \sqrt{a^2 - b}. \qquad \cdots \qquad \cdots \qquad (iv)$

$\therefore$ Adding (iii) and (iv) and subtracting (iv) from (iii) and dividing by 2, we get

$$x = \tfrac{1}{2}\{a + \sqrt{a^2 - b}\}; \text{ and } y = \tfrac{1}{2}\{a - \sqrt{a^2 - b}\}.$$

Hence the required root is

$$\pm [\sqrt{\{\tfrac{1}{2}a + \tfrac{1}{2}\sqrt{(a^2 - b)}\}} + \sqrt{\{\tfrac{1}{2}a - \tfrac{1}{2}\sqrt{(a^2 - b)}\}}].$$

Cor. Similarly the square root of $a - \sqrt{b}$ can be obtained ; in this case *we should assume* $\sqrt{(a - \sqrt{b})} = \sqrt{x} - \sqrt{y}.$

Note 1. If $a^2 - b$ be a positive perfect square, and a is positive then x and y are rational and the square root of $a + \sqrt{b}$ becomes a quadratic surd. If $a^2 - b$ be not a perfect square, then the calculation of the square root by the above method becomes complicated. In such cases, we may proceed in Ex. 2 below.

Note 2. The square root of a binomial surd can often be found by inspection, as shown in Ex. 2 below.

To find the square root of $a + \sqrt{b} + \sqrt{c} + \sqrt{d}$.

Let $\sqrt{(a + \sqrt{b} + \sqrt{c} + \sqrt{d})} = \sqrt{x} + \sqrt{y} + \sqrt{z}.$

$\therefore$ Squaring both sides, we get

$$a + \sqrt{b} + \sqrt{c} + \sqrt{d} = x + y + z + 2\sqrt{xy} + 2\sqrt{yz} + 2\sqrt{zx}.$$

Then we assume

$$2\sqrt{xy} = \sqrt{b}, \; 2\sqrt{yz} = \sqrt{c}, \; 2\sqrt{xz} = \sqrt{d}, \text{ and } x + y + z = a.$$

If the values of x, y, z found from the first three equations also satisfy $x+y+z=a$, then the required square root has been found, but not otherwise, the condition for which is obviously

$$bc+cd+db=2a\sqrt{bcd}.$$

20. Illustrative Examples.

Ex. 1. *Find the square root of* $\frac{1}{2}(2+\sqrt{3})$. [*C. U. 1924*]

$$\tfrac{1}{2}(2+\sqrt{3})=\tfrac{1}{4}(4+2\sqrt{3})=\tfrac{1}{4}(1+3+2\sqrt{3})=\tfrac{1}{4}(1+\sqrt{3})^2.$$

$\therefore$ The reqd. square root $=\pm\frac{1}{2}(1+\sqrt{3})$.

Ex. 2. *Find the square root of* $6+4\sqrt{3}$.

Here $a^2-b=36-48=-12$, which is not a perfect square. Hence the method of Art. 19 is not suitable. We proceed as follows :

$$6+4\sqrt{3}=\sqrt{3}(4+2\sqrt{3})=\sqrt{3}(3+1+2\sqrt{3})=\sqrt{3}(\sqrt{3}+1)^2.$$

$\therefore$ The reqd. square root $=\pm\sqrt[4]{3}(\sqrt{3}+1)=\pm(\sqrt[4]{27}+\sqrt[4]{3})$.

Ex. 3. *Find the square root of* $\sqrt{50}+\sqrt{48}$.

Here, $\quad \sqrt{50}+\sqrt{48}=\sqrt{2}(\sqrt{25}+\sqrt{24})=\sqrt{2}(5+2\sqrt{6})$
$$=2\{3+2+2\sqrt{(2\times3)}\}=\sqrt{2}(\sqrt{3}+2)^2.$$

$\therefore$ The reqd. square root $=\pm\sqrt[4]{2}(\sqrt{3}+\sqrt{2})$.

Ex. 4. *Find the square root of* $11+6\sqrt{2}+4\sqrt{3}+2\sqrt{6}$.

Let $\quad \sqrt{11+6\sqrt{2}+4\sqrt{3}+2\sqrt{6}}=\sqrt{x}+\sqrt{y}+\sqrt{z}$.

Squaring both sides,

$$11+6\sqrt{2}+4\sqrt{3}+2\sqrt{6}=x+y+z+2(\sqrt{xy}+\sqrt{yz}+\sqrt{zx}).$$

This is satisfied if $x+y+z=11$, $2\sqrt{xy}=6\sqrt{2}$, $2\sqrt{yz}=4\sqrt{3}$, $2\sqrt{zx}=2\sqrt{6}$; hence we have, $xy=18$, $yz=12$, $zx=6$, whence $xyz=36$.

$\therefore$ The reqd. square root $=\sqrt{3}+\sqrt{6}+\sqrt{2}$.

Ex. 5. *If* $x=\dfrac{\sqrt{a+2b}+\sqrt{a-2b}}{\sqrt{a+2b}-\sqrt{a-2b}}$, *show that* $bx^2-ax+b=0$.

[*C. U. 1935*]

By Componendo, we have

$$\frac{x+1}{x-1}=\frac{\sqrt{a+2b}}{\sqrt{a-2b}}, \quad \text{or,} \quad \frac{x^2+2x+1}{x^2-2x+1}=\frac{a+2b}{a-2b}.$$

Again, by Componendo and Dividendo,

$$\frac{2(x^2+1)}{4x}=\frac{2a}{4b}, \text{ or, } \frac{x^2+1}{x}=\frac{a}{b}. \quad \therefore \ bx^2-ax+b=0.$$

Ex. 6. *If* $x=2+2^{\frac{2}{3}}+2^{\frac{1}{3}}$, *find the value of*

$$x^3-6x^2+6x-2.$$

We have, $\quad x-2=2^{\frac{2}{3}}+2^{\frac{1}{3}}=2^{\frac{1}{3}}\left(2^{\frac{1}{3}}+1\right).$

Cubing both sides,

$$x^3-6x^2+12x-8=2\left(1+2^{\frac{1}{3}}\right)^3=2\left(1+3.2^{\frac{1}{3}}+3.2^{\frac{2}{3}}+2\right).$$

$$\therefore \ x^3-6x^2+6x-2=(x^3-6x^2+12x-8)-6x+6$$

$$=2\left(3+3.2^{\frac{1}{3}}+3.2^{\frac{2}{3}}\right)-6x+6$$

$$=6\left(1+2^{\frac{1}{3}}+2^{\frac{2}{3}}\right)-6x+6$$

$$=6\left(2+2^{\frac{2}{3}}+2^{\frac{1}{3}}-x\right)=6\ (x-x)=0.$$

Ex. 7. *Find the square root of* $\frac{1}{3}(3x-4)+\frac{2}{3}\sqrt{2x^2-3x-5}.$

The expression $=\frac{1}{3}\{(2x-5)+(x+1)+2\sqrt{(2x-5)(x+1)}\}.$

Putting $\quad a=\sqrt{2x-5}$ and $b=\sqrt{x+1}$,

we find the expression $=\frac{1}{3}\ (a^2+b^2+2ab)=\frac{1}{3}\ (a+b)^2.$

$\therefore$ The reqd. square root $=\dfrac{1}{\sqrt{3}}\ (a+b)=\dfrac{1}{\sqrt{3}}\left(\sqrt{2x-5}+\sqrt{x+1}\right).$

Ex. 8. *Given* $\sqrt{5}=2\cdot236$, *find the value of*

$$\frac{\sqrt{(3+\sqrt{5})}}{\sqrt{2}-\sqrt{(7-3\sqrt{5})}}.$$

Multiplying numerator and denominator by $\sqrt{2}$,

the expression $=\dfrac{\sqrt{(6+2\sqrt{5})}}{2-\sqrt{(14-6\sqrt{5})}}.$

Now, $6+2\sqrt{5}=(\sqrt{5})^2+1+2\sqrt{5}=(\sqrt{5}+1)^2,$

and $14-6\sqrt{5}=3^2+(\sqrt{5})^2-2.3\sqrt{5}=(3-\sqrt{5})^2.$

$\therefore$ The exp. $=\dfrac{\sqrt{5}+1}{2-(3-\sqrt{5})}=\dfrac{\sqrt{5}+1}{\sqrt{5}-1}=\dfrac{(\sqrt{5}+1)^2}{5-1}=\dfrac{5+2\sqrt{5}}{4}$

$$=\frac{3+\sqrt{5}}{2}=\frac{5\cdot236}{2}=2\cdot618.$$

Ex. 9. *Find the cube root of* $7 + 5\sqrt{2}$.

Let $\quad\sqrt[3]{(7 + 5\sqrt{2})} = x + \sqrt{y},$ $\qquad\cdots$ (1)

then $\quad\sqrt[3]{(7 - 5\sqrt{2})} = x - \sqrt{y}.$ $\qquad\cdots$ (2)

Multiplying (1) and (2), $x^2 - y = \sqrt[3]{(49 - 50)}$

$$= \sqrt[3]{(-1)} = -1. \qquad \cdots \text{(3)}$$

Cubing (1), $x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y} = 7 + 5\sqrt{2}.$

$$x^3 + 3xy = 7.$$

Substituting the value of y from (3) in (4), we have

$$x^3 + 3x\,(x^2 + 1) = 7,$$

or, $\quad 4x^3 + 3x = 4 + 3,$

or, $\quad 4\,(x^3 - 1) + 3\,(x - 1) = 0, \quad i.e., \quad (x-1)(4x^2 + 4x + 7) = 0.$

$\therefore \quad x - 1 = 0$, whence $x = 1$;

or, $\quad 4x^2 + 4x + 7 = 0$, which gives imaginary values of x.

[See *Art. 35*]

$\therefore \quad$ Taking $x = 1$, we have $y = 2$ from (3).

Hence, the reqd. cube root is $1 + \sqrt{2}$.

Note. *The cube root of an expression of the form* $a \pm \sqrt{b}$ *may sometimes be found as above. The above-method is useless unless* $x^2 - y$ *is rational.* If we have considered the two imaginary values of x, we would have altogether obtained *three cube roots* of the above expression, one real and two imaginary. Here the real cube root is implied.

Examples IV

1. Find the square roots of :

(i) $4 + 2\sqrt{3}$. $\qquad\qquad$ (ii) $16 - 5\sqrt{7}$.

(iii) $28 - 5\sqrt{12}$. $\qquad\qquad$ (iv) $10 + 6\sqrt{5}$.

(v) $\sqrt{45} + \sqrt{40}$. $\qquad\qquad$ (vi) $3\sqrt{6} - 4\sqrt{3}$.

(vii) $10 + 2\sqrt{6} + 2\sqrt{15} + 2\sqrt{10}$.

(viii) $a + b + c + 2\sqrt{(bc + ca)}$.

(ix) $\frac{1}{2}(3x+2) + \sqrt{2x^2+5x-3}$.

(x) $5 - \sqrt{10} - \sqrt{15} + \sqrt{6}$.

(xi) $1 + x^2 + \sqrt{(1 + x^2 + x^4)}$.

(xii) $a + b + \sqrt{a^2 + 2ab}$.

(xiii) $\sqrt{(b-c)(c-a)} + \sqrt{(c-a)(a-b)} + \sqrt{(a-b)(b-c)}$.

2. Shew that

(i) $\dfrac{\sqrt{(5 + 2\sqrt{6})} - \sqrt{(5 - 2\sqrt{6})}}{\sqrt{(5 + 2\sqrt{6})} + \sqrt{(5 - 2\sqrt{6})}} = \sqrt{\dfrac{2}{3}}$.

(ii) $\dfrac{1}{\sqrt{(11 - 2\sqrt{30})}} - \dfrac{3}{\sqrt{(7 - 2\sqrt{10})}} - \dfrac{4}{\sqrt{(8 + 4\sqrt{3})}} = 3$.

3. Simplify

(i) $\dfrac{2 + \sqrt{3}}{\sqrt{2} + \sqrt{(2 + \sqrt{3})}} + \dfrac{2 - \sqrt{3}}{\sqrt{2} - \sqrt{(2 - \sqrt{3})}}$.

(ii) $\dfrac{1 + x}{1 + \sqrt{1+x}} + \dfrac{1 - x}{1 - \sqrt{1-x}}$, where $x = \dfrac{\sqrt{3}}{2}$.

4. (a) Find the rationalising factors of

    (i) $\sqrt{p} + \sqrt{q} + \sqrt{r}$,   (ii) $\sqrt[3]{3} - \sqrt{2}$.

    (iii) $\sqrt[3]{4} - \sqrt[3]{2} + 1$.

(b) Express with a rational denominator

    (i) $\dfrac{\sqrt[3]{2} - 1}{\sqrt[3]{2} + 1}$.   (ii) $\dfrac{2 - \sqrt[3]{3}}{2 + \sqrt[3]{3}}$.

    (iii) $\dfrac{3}{\sqrt[3]{4} + \sqrt[3]{2} + 1}$.   (iv) $\dfrac{\sqrt{3}}{\sqrt{2} + \sqrt{3} - \sqrt{5}}$.

5. If $x = 2 + \sqrt{3}$, show that $x^3 - 2x^2 - 7x + 2 = 0$.

6. If $x = 1 + \sqrt{2} + \sqrt{3}$, find the value of

$$2x^4 - 8x^3 - 5x^2 + 26x - 28.$$

7. If $x = \dfrac{\sqrt{3}+1}{\sqrt{3}-1}$ and $y = \dfrac{\sqrt{2}-1}{\sqrt{3}+1}$, show that

(i) $\dfrac{x^2 + xy + y^2}{x^2 - xy + y^2} = \dfrac{15}{13}$.

(ii) $x^4 + x^2 y^2 + y^4 = 195$.

8. Show that $(2 + \sqrt{3})^{\frac{3}{2}} + (2 - \sqrt{3})^{\frac{3}{2}} = 3\sqrt{6}$.

9. (a) Find the square root of $\dfrac{\sqrt{5}+1}{\sqrt{5}-1}$ correct to three places of decimals.

(b) Given $\sqrt{3} = 1\cdot732$, find the value of

$$\dfrac{\sqrt{(2 + \sqrt{3})}}{\sqrt{2} - \sqrt{(6 - 3\sqrt{3})}}.$$

10. Find the cube roots of :

(i) $10 + 6\sqrt{3}$.　　(ii) $22 - 10\sqrt{7}$.　　(iii) $9\sqrt{3} + 11\sqrt{2}$.

11. If $x = \sqrt[3]{7 + 5\sqrt{2}} + \sqrt[3]{7 - 5\sqrt{2}}$, then

$$x^3 + 3x - 14 = 0.$$

12. If $x = \dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} + \sqrt{2}}$ and $y = \dfrac{\sqrt{3} + \sqrt{2}}{\sqrt{3} - \sqrt{2}}$, show that

$$3x^2 - 5xy + 3y^2 = 289.$$

13. Prove that

$$\sqrt{-\sqrt{3} + \sqrt{3 + 8\sqrt{7 + 4\sqrt{3}}}} = 2.$$

14. (i) If $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}} = 0$, then $(x + y + z)^3 - 27xyz = 0$.

(ii) If $\sqrt[3]{(a+b)} + \sqrt[3]{(b+c)} + \sqrt[3]{(c+a)} = 0$, show that

$$(a + b + c)^3 = 9(a^3 + b^3 + c^3).$$

15. Show that $\sqrt{[a\sqrt{\{a\sqrt{(a\ldots\ldots \text{ to infinity})\}}}]} = a$.

16. If $(x + \sqrt{x^2 - bc})(y + \sqrt{y^2 - ca})(z + \sqrt{z^2 - ab})$
$$= (x - \sqrt{x^2 - bc})(y - \sqrt{y^2 - ca})(z - \sqrt{z^2 - ab}),$$

show that each of the expressions $= \pm abc$.

17. If $\sqrt{x} + \sqrt{(a-x)} = \sqrt{y} + \sqrt{(a-y)} = \sqrt{z} + \sqrt{(a-z)}$,
then $(y-z)(z-x)(x-y) = 0$.

18. Find the value of $\dfrac{x^2 + xy + y^2}{x^2 - xy + y^2}$, when

$$x = \frac{\sqrt{a+1} + \sqrt{a-1}}{\sqrt{a+1} - \sqrt{a-1}} \quad \text{and} \quad y = \frac{\sqrt{a+1} - \sqrt{a-1}}{\sqrt{a+1} + \sqrt{a-1}}.$$

19. (i) Rationalise the equation $\sqrt{x} \pm \sqrt{y} \pm \sqrt{z} = 0$.
 (ii) If $\sqrt{(y-z)} + \sqrt{(z-x)} + \sqrt{(x-y)} = 0$,
 then $x = y = z$.

20. If $x = z\sqrt{(1-y^2)} + y\sqrt{(1-z^2)}$, then
$(x+y+z)(x-y+z)(x+y-z)(y+z-x) = 4x^2y^2z^2$.

ANSWERS

1. (i) $(\sqrt{3}+1)$.   (ii) $\dfrac{5-\sqrt{7}}{\sqrt{2}}$.   (iii) $5-\sqrt{3}$.

(iv) $5^{\frac{1}{4}}(\sqrt{5}+1)$.   (v) $5^{\frac{1}{4}}(\sqrt{2}+1)$.   (vi) $6^{\frac{1}{4}}(\sqrt{2}-1)$.
(vii) $(\sqrt{2}+\sqrt{3}+\sqrt{5})$.   (viii) $\{\sqrt{(a+b)}+\sqrt{c}\}$.

(ix) $\dfrac{1}{\sqrt{2}}(\sqrt{2x-1}+\sqrt{x+3})$.   (x) $\dfrac{1}{\sqrt{2}}(\sqrt{3}+\sqrt{2}-\sqrt{5})$.

(xi) $\dfrac{1}{\sqrt{2}}(\sqrt{1+x+x^2}-\sqrt{1-x+x^2})$.   (xi) $\dfrac{1}{\sqrt{2}}(\sqrt{a}+\sqrt{a+2b})$.

(xiii) $\dfrac{1}{\sqrt{2}}(\sqrt{b-c}+\sqrt{c-a}+\sqrt{a-b})$.

3. (i) $\sqrt{2}$.   (ii) 1.

4. (a) (i) $(\sqrt{p}-\sqrt{q}+\sqrt{r})(\sqrt{p}+\sqrt{q}-\sqrt{r})(-\sqrt{p}+\sqrt{q}+\sqrt{r})$.
 (ii) $3\sqrt[3]{9}+3\sqrt[3]{3}\sqrt{2}+6+2\sqrt[3]{2}\sqrt{2}+4\sqrt[3]{3}+4\sqrt{2}$.   (iii) $\sqrt[3]{2}+1$.

(b) (i) $\dfrac{1-2^{\frac{5}{3}}+2^{\frac{4}{3}}}{3}$.   (ii) $\dfrac{5-8\sqrt[3]{3}+4\sqrt[3]{9}}{11}$.

(iii) $3(2^{\frac{1}{3}}-1)$.   (iv) $\dfrac{2+\sqrt{6}+\sqrt{10}}{4}$.

6. $6\sqrt{6}$.   **9.** (a) $1{\cdot}618$.   (b) $3{\cdot}732$.

10. (i) $\sqrt{3}+1$.   (ii) $1-\sqrt{7}$.   (iii) $\sqrt{3}+\sqrt{2}$.

18. $\dfrac{4a^2-1}{4a^2-3}$.   **19.** (i) $x^2+y^2+z^2-2(xy+yz+zx)=0$.

IMAGINARY QUANTITIES

21. Definition.

Since all squares whether of positive or negative quantities are positive, it follows that the square root of a negative quantity $\sqrt{-a^2}$ cannot represent any positive or negative quantity *i.e.*, $\sqrt{-a^2}$ can never be equal to $+a$ or $-a$. Such a quantity on this account is called an *Imaginary quantity*, as distinct from real quantities, so far dealt with.

Here the word 'imaginary' is used not in the ordinary sense of the word ; in fact an imaginary quantity has as much real existence as a real number itself.

As $\sqrt{a}$ is defined to be a quantity such that $(\sqrt{a})^2 = a$. so also $\sqrt{-a}$ is defined to be a quantity such that $(\sqrt{-a})^2$ *i.e.*, $\sqrt{-a} \times \sqrt{-a} = -a$.

Similarly, $\sqrt{-1}$ denotes a quantity such that

$$(\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = -1.$$

Note. The fundamental laws of Algebra, such as the Associative, Distributive and Commutative, are assumed to hold in the case of imaginary quantities also.

22. Meaning of the symbol *i*.

We know that -1 denotes an operation which performed upon any quantity changes its sign. If we suppose that $\sqrt{-1}$ obeys the law expressed by $\sqrt{-1} \times \sqrt{-1} = -1$, it follows that $\sqrt{-1}$ must be regarded as an operation which when repeated twice is equivalent to a reversal in sign. Since any magnitude whatever can be represented by lengths set off along a straight line, we may consider that the operation $\sqrt{-1}$ performed twice in the same direction in succession rotates the line directly opposite to its original

direction, that is, turns the line through two right angles. It follows therefore, that the operation $\sqrt{-1}$ may be considered to be a revolution through a right angle, and $\sqrt{-1}$, as a symbol of operation, has a perfectly definite meaning and for shortness it is denoted by i. The operation denoted by i is considered to be a revolution through a right angle counter-clockwise, $-i$ denoting revolution through a right angle in the opposite direction.

Again a units of length rotated through a right angle counter-clockwise give the same result as a unit length rotated through a right angle counter-clockwise and then multiplied by a. Thus, $ai = ia$.

Also to multiply ai and bi is to do to ai what is done to unit to obtain bi, $i.e.$, we must multiply by b and then rotate through two right angles, so that $ai \times bi = -ab = abii$.

Since $(ai) \times (ai) = aaii = a^2(-1) = -a^2$, it follows that $\sqrt{-a^2} = ai = a \times \sqrt{-1}$.

Thus, an imaginary quantity is equivalent to the product of a real quantity and $\sqrt{-1}$. Hence, in all our investigations it is therefore necessary to use only one imaginary expression namely $\sqrt{-1}$, which for shortness is usually denoted by the symbol i.

Thus, $i^2 = -1$; as i is subject to the general laws of Algebra, we get

(i) $7i + 4i = 11i$.   (ii) $7i - 4i = 3i$.

(iii) $7i \times 4i = 28i^2 = -28$.   (iv) $7i \div 4i = \frac{7}{4}$.

23. Powers of i.

$$i^2 = -1 ; \quad i^3 = i^2.i = -i ; \quad i^4 = (i^2)^2 = (-1)^2 = 1 ;$$

and generally, if n is any positive integer,

$$i^{4n} = (i^4)^n = 1 ; \qquad i^{4n+2} = i^{4n}.i^2 = -1 ;$$
$$i^{4n+1} = i^{4n}.i = i ; \qquad i^{4n+3} = i^{4n}.i^3 = -i.$$

Thus, *positive integral powers of i reduce only to one of the four values* ± 1 *and* $\pm i$.

Again, $i^{-1} = \dfrac{1}{i} = \dfrac{-i^2}{i} = -i$; $i^{-2} = \dfrac{1}{i^2} = -1$;

$\qquad i^{-3} = i^{-2} . i^{-1} = -1 \times (i) = i$;

$\qquad i^{-4} = (i^{-2})^2 = 1$; etc.

Hence, *negative integral powers of i also reduce only to one of the four values* ± 1 *and* $\pm i$.

24. Complex Quantity.

A pair of real numbers (a, b) united symbolically in the form $a + ib$ is called a *complex quantity*. A complex quantity, thus consists of a real part *viz.*, a and an imaginary part *viz.*, ib. It reduces to a purely real or a purely imaginary quantity, according as $b = 0$, or, $a = 0$.

When two complex quantities differ only in the sign of the imaginary parts, they are called *conjugate complex quantities*. Thus, $a + ib$ and $a - ib$ are conjugate complex quantities.

25. Properties of Complex Quantities.

(i) **If a + ib = 0, then a = 0, b = 0.**

Since $\qquad a + ib = 0$; $\qquad \therefore \quad a = -ib.$

$\therefore$ Squaring, $a^2 = -b^2$, or, $a^2 + b^2 = 0$.

Since a^2 and b^2 are both positive, their sum cannot be zero, unless each is separately zero. Hece, $a = b = 0$.

(ii) **If a + ib = c + id, then a = c, b = d.**

By transposition, $(a - c) = -i(b - d)$.

$\therefore$ Squaring, $(a - c)^2 = -(b - d)^2$,

$\qquad$ or, $(a - c)^2 + (b - d)^2 = 0$.

Since, $(a - c)^2$ and $(b - d)^2$ are both positive, their sum cannot be zero, unless each separately be zero. Hence $a - c = 0$ and $b - d = 0$, *i.e.*, $a = c$ and $b = d$

(iii) *The sum and the product of two conjugate complex quantities are both real.*

Thus, $(a+ib)+(a-ib)=2a$ (real)
$$(a+ib)(a-ib)=a^2-i^2b^2=a^2+b^2 \text{ (real)}.$$

(iv) *The sum or difference of two complex quantities is a complex quantity.*

Thus, $(a+ib)\pm(c+id)=(a\pm c)+i(b\pm d)=A+iB$, say.

Similarly, $(a+ib)\pm(c+id)\pm(e+if)=(A+iB)\pm(e+if)=C+iD$, say, as before ; and so on.

(v) *The product of two or more complex quantities is a complex quantity.*

$$\begin{aligned}(a+ib)(c+id)&=ac+iad+ibc+i^2bd\\&=(ac-bd)+i(bc+ad), \text{ since } i^2=-1\\&=a+i\beta, \text{ say.}\end{aligned}$$

$$\begin{aligned}\therefore \ (a+ib)(c+id)(e+if)&=(a+i\beta)(e+if)\\&=(ae-\beta f)+i(\beta e+af), \text{ as before,}\end{aligned}$$

which is evidently of the form $A+iB$, and so on, for any number of factors.

(vi) *The quotient of two complex quantities is a complex quantity.*

$$\frac{a+ib}{c+id}=\frac{(a+ib)(c-id)}{(c+id)(c-id)}=\frac{ac-iad+ibc-i^2bd}{c^2-i^2d^2}$$

$$=\frac{(ac+bd)+i(bc-ad)}{c^2+d^2}=\frac{ac+bd}{c^2+d^2}+i\frac{bc-ad}{c^2+d^2},$$

which is of the form $A+iB$.

(vii) *Any positive integral power of a complex quantity is a complex quantity.*

$$(a+ib)^2=a^2-b^2+i2ab$$

$$(a+ib)^3=(a^3-3ab^2)+i(3a^2b-b^3)$$

both of which are of the form $A+iB$, and similarly for any other positive integral power of $a+ib$.

(viii) *Any root of a complex quantity is a complex quantity.*

Let x be equal to the nth root of $a + ib$,

$$\text{then } \sqrt[n]{(a+ib)} = x. \qquad \therefore \quad a + ib = x^n.$$

Now, if x be real x^n is real, which is impossible, since in that case b would be zero. Hence, x is a complex quantity.

26.   Square Root of a + ib.

Since, any root of a complex quantity is a complex quantity, let us assume

$$\sqrt{a+ib} = x + iy, \text{ where } x \text{ and } y \text{ are real}$$

$$\text{then, } a + ib = (x+iy)^2 = x^2 - y^2 + 2ixy.$$

$\therefore$ Equating real and imaginary parts, we get

$$\therefore \quad x^2 - y^2 = a \qquad \cdots \quad (1) \quad [\text{ See } Art.\ 25(ii)\,]$$

$$2xy = b \qquad \cdots \quad (2)$$

Now, $(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2 y^2 = a^2 + b^2$;

$$\therefore \quad x^2 + y^2 = \sqrt{a^2 + b^2}. \quad \cdots \quad (3)^*$$

From (1) and (3) by adding and subtracting, we get

$$\therefore \quad x^2 = \tfrac{1}{2}\left(\sqrt{a^2+b^2} + a\right) \text{ and } y^2 = \tfrac{1}{2}\left(\sqrt{a^2+b^2} - a\right) ;$$

$$\therefore \quad x = \pm\left\{\tfrac{1}{2}\left(\sqrt{a^2+b^2} + a\right)\right\}^{\frac{1}{2}}$$

$$\text{and} \quad y = \pm\left\{\tfrac{1}{2}\left(\sqrt{a^2+b^2} - a\right)\right\}^{\frac{1}{2}}.$$

From (2), it is clear that xy must have the same sign as b and hence (i) if b be *positive*, both x and y have the same sign, positive or negative and (ii) if b be *negative*, x and y have opposite signs.

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* [Since, x and y are real, $x^2 + y^2$ is positive, and hence in (3) we take only the positive value of the square root.]

Hence, when b is *positive*, the required square roots are

$$\pm\left[\{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)\}^{\frac{1}{2}}+i\{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)\}^{\frac{1}{2}}\right]$$

and when b is *negative*, the required square roots are

$$\pm\left[\{\tfrac{1}{2}(\sqrt{a^2+b^2}+a)\}^{\frac{1}{2}}-i\{\tfrac{1}{2}(\sqrt{a^2+b^2}-a)\}^{\frac{1}{2}}\right].$$

Note 1. It is clear from above that a complex quantity has two square roots which differ only in signs.

Note 2. The square root of a complex quantity can sometimes be found by inspection ; it is shown in Ex. 2 and Ex. 3 below.

Ex. 1. *Find the square root of* $5-i.12$.

Let $\sqrt{5-i.12}=x+iy$; then $5-i.12=x^2-y^2+2ixy$.

$$\therefore\quad x^2-y^2=5\qquad\cdots\qquad\cdots\qquad(1)$$
$$2xy=-12.\qquad\cdots\qquad\cdots\qquad(2)$$
$$\therefore (x^2-y^2)^2=(x^2-y^2)^2+(2xy)^2=25+144=169.$$
$$\therefore\quad x^2+y^2=13.\qquad\cdots\qquad\cdots\qquad(3)$$

From (1) and (3), $x^2=9$ and $y^2=4$; $\therefore\quad x=\pm3, y=\pm2$.

Since, xy is negative, therefore, x and y must be of opposite signs.

$$\therefore\quad x=3, y=-2,\quad\text{or,}\quad x=-3, y=2.$$

Hence, the required square roots are $3-2i$ and $-3+2i$, *i.e.*, $\pm(3-2i)$.

Ex. 2. *Find the square roots of* i *and* $-i$.

(i) $i=\tfrac{1}{2}(1+2i-1)=\tfrac{1}{2}(1+2i+i^2)=\tfrac{1}{2}(1+i)^2$;

$$\therefore\qquad\sqrt{i}=\pm\frac{1}{\sqrt{2}}(1+i).$$

(ii) $-i=\tfrac{1}{2}(1-2i-1)=\tfrac{1}{2}(1-2i+i^2)=\tfrac{1}{2}(1-i)^2$;

$$\therefore\quad\sqrt{-i}=\pm\frac{1}{\sqrt{2}}(1-i).$$

Ex. 3. *Find the square root of* $\tfrac{1}{2}(-1+\sqrt{-3})$.

$$\tfrac{1}{2}(-1+\sqrt{-3})=\tfrac{1}{4}(-2+2\sqrt{-3})=\tfrac{1}{4}(-3+1+2\sqrt{-3})$$
$$=\tfrac{1}{4}(\sqrt{-3}+1)^2.$$

$\therefore$ The required square roots $=\pm\tfrac{1}{2}(\sqrt{-3}+1)$.

Ex. 4. *Find the square root of* $\dfrac{5 + i.12}{3 - i.4}$.

The given exp. $= \dfrac{(5 + i.12)(3 + i4)}{(3 - i4)(3 + i4)} = \dfrac{-33 + i56}{25}$.

Now, the square root of the numerator $= \pm (4 + i7)$ obtained by the method of Ex. 1 above ; or it can be found from the relation $-13 + i.56 = -49 + 16 + i.56 = (7i)^2 + 4^2 + 2.7i.4 = (4 + i7)^2$.

Hence, the required square roots $= \pm \frac{1}{5} (4 + i7)$.

27. Modulus of a complex quantity.

The positive value of the square root of $a^2 + b^2$ *i.e.*, $+ \sqrt{a^2 + b^2}$ is called the *modulus* of the complex quantity $a + ib$. Thus, the modulus of $3 + 4i = \sqrt{3^2 + 4^2} = 5$.

The modulus of $a + ib$ is shortly written as mod $(a + ib)$. Thus, mod $(a + ib) = + \sqrt{a^2 + b^2}$.

28. Properties of the Modulus.

(i) *A complex quantity and its conjugate have the same modulus.*

Thus, the modulus of $a - ib = \sqrt{\{a^2 + (-b)^2\}} = + \sqrt{a^2 + b^2}$ which is the modulus of $a + ib$. Since $(a + ib)(a - ib) = a^2 - b^2$, the modulus of either of two conjugate complex, expressions is equal to the positive square root of their product.

(ii) *The modulus of the product of two complex quantities is equal to the product of their moduli.*

Let $a + ib$ and $c + id$ be two complex quantities, then

$$(a + ib)(c + id) = (ac - bd) + i (bc + ad).$$

$\therefore$ modulus of the product

$$= \sqrt{(ac - bd)^2 + (bc + ad)^2}$$
$$= \sqrt{(a^2 c^2 + b^2 d^2 + b^2 c^2 + a^2 d^2)}$$
$$= \sqrt{(a^2 + b^2)(c^2 + d^2)}$$
$$= \sqrt{(a^2 + b^2)} . \sqrt{c^2 + d^2}$$
$$= \text{mod} (a + ib) \times \text{mod} (c + id).$$

(iii) *The modulus of the quotient of two complex quantities is the quotient of their moduli.*

Let $a+ib$ and $c+id$ be two complex quantities.

Now, $\dfrac{a+ib}{c+id} = \dfrac{(a+ib)(c-id)}{(c+id)(c-id)} = \dfrac{(ac+bd)+i\,(bc-ad)}{c^2+d^2}$

$$= \dfrac{ac+bd}{c^2-d^2} + i\dfrac{bc-ad}{c^2+d^2}.$$

$\therefore$ The modulus of the quotient

$$= \sqrt{\left\{ \left(\dfrac{ac+bd}{c^2+d^2}\right)^2 + \left(\dfrac{bc-ad}{c^2+d^2}\right)^2 \right\}}$$

$$= \sqrt{\left\{ \dfrac{a^2c^2+b^2d^2+b^2c^2+a^2d^2}{(c^2+d^2)^2} \right\}}$$

$$= \sqrt{\left\{ \dfrac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2} \right\}}$$

$$= \dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}} = \dfrac{\mathrm{mod}\,(a+ib)}{\mathrm{mod}\,(c+id)}.$$

29. Cube roots of Unity.

Let $x = \sqrt[3]{1}$; then $x^3 = 1$,   or, $x^3 - 1 = 0$.

$\therefore$ $(x-1)(x^2+x+1) = 0$.

$\therefore$ Either, $x - 1 = 0$, whence $x = 1$,

or, $x^2 + x + 1 = 0$, whence $x = \tfrac{1}{2}(-1 \pm \sqrt{-3})$.

[See *Art. 35*]

Thus, there are *three cube roots of unity viz.*,

$$1, \tfrac{1}{2}(-1+\sqrt{-3}) \text{ and } \tfrac{1}{2}(-1-\sqrt{-3}),$$

of which the first is real and the other two complex.

30. Properties of Cube Roots of Unity.

(i) *Each imaginary cube root of unity is the square of the other.*

$$\{\tfrac{1}{2}(-1+\sqrt{-3})\}^2 = \tfrac{1}{4}(1-3-2\sqrt{-3}) = \tfrac{1}{2}(-1-\sqrt{-3}).$$

$$\{\tfrac{1}{2}(-1-\sqrt{-3})\}^2 = \tfrac{1}{4}(1-3+2\sqrt{-3}) = \tfrac{1}{2}(-1+\sqrt{-3}).$$

(ii) *The product of two imaginary cube roots of unity is unity ; i.e., either of the two roots is the reciprocal of the other.*

$$\{\tfrac{1}{2}(-1+\sqrt{-3})\}\{\tfrac{1}{2}(-1-\sqrt{-3})\}=\tfrac{1}{4}\{1-(\sqrt{-3})^2\}$$
$$=\tfrac{1}{4}\{1+3\}-1.$$

(iii) *The sum of the three cube roots of unity is zero.*
$$1+\tfrac{1}{2}\{-1+\sqrt{-3}\}+\tfrac{1}{2}\{-1-\sqrt{-3}\}=1-1=0.$$

Note 1. Since each imaginary cube root is the square of the other, if ω denote either of the two imaginary cube roots, the other is ω^2 ; hence, *the three cube roots of unity are usually denoted by* 1, ω, ω^2. With this notation, results of (ii) and (iii) become $\omega.\omega^2$, *i.e.*, $\omega^3=1$ and $1+\omega+\omega^2=0$. These results can also be established thus :—

Since, ω is a root of $x^2+x+1=0$, $\therefore$ $\omega^2+\omega+1=0$.

Since, ω is a root of $x^3-1=0$. $\therefore$ $\omega^3=1$, *i.e.*, $\omega.\omega^2=1$.

Note 2. The cube root of every real quantity has three values, one of which is real (*i.e.*, the arithmetical cube root) and the other two imaginary and these three roots can be obtained by multiplying the arithmetical cube root by ω, ω^2.

31. Powers of ω.

Any positive integral power of ω is equal to 1, ω, *or* ω^2.

Thus, $\omega^4=\omega^3.\omega=\omega$; $\omega^5=\omega^3.\omega^2=\omega^2$; $\omega^6=(\omega^3)^2=1$; etc.

Generally, if n is any positive integer,

and if $n=3p$ (p being an integer), $\omega^n=\omega^{3p}=(\omega^3)^p=1$

if $n=3p+1$, $\omega^n=\omega^{3p+1}=\omega^{3p}.\omega=\omega$

if $n=3p+2$, $\omega^n=\omega^{3p+2}=\omega^{3p}.\omega^2=\omega^2$.

Thus, $\omega^n=1$, ω or ω^2, according as n, when divided by 3 leaves 0, 1 or 2 as remainder.

Similarly, it can be shown that *any negative integral power of ω is equivalent to* 1, ω *or* ω^2.

4

32. Illustrative Examples.

Ex. 1. *Find the cube roots of* -1.

Let $x = \sqrt[3]{-1}$; then $x^3 = -1$; $\therefore x^3 + 1 = 0$.

$\therefore (x+1)(x^2 - x + 1) = 0$.

$\therefore$ either, $x + 1 = 0$, whence $x = -1$,

or, $x^2 - x + 1 = 0$, whence $x = \frac{1}{2}(1 \pm \sqrt{-3})$.

Thus, the cube roots are -1, $\frac{1}{2}(1 + \sqrt{-3})$ and $\frac{1}{2}(1 - \sqrt{-3})$.

Ex. 2. *Find the fourth roots of* 1.

Let $x = \sqrt[4]{1}$; then $x^4 = 1$; $\therefore x^4 - 1 = 0$.

$\therefore (x^2 + 1)(x^2 - 1) = 0$,

i.e., $(x+i)(x-i)(x+1)(x-1) = 0$.

Thus, the four fourth roots of 1 are i, $-i$, 1, -1.

Ex. 3. *Show that*

(i) $(\omega a + \omega^2 b)(\omega^2 a + \omega b) = a^2 - ab + b^2$.   [*C. U. 1929*]

(ii) $(a - \omega b)(a - \omega^2 b) = a^2 + ab + b^2$.

(iii) $(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c) = a^2 + b^2 + c^2 - bc - ca - ab$.

The above three results follow by actual multiplication and by use of the relations

$$\omega + \omega^3 = -1, \ \omega^2 + \omega^4 = \omega^2 + \omega = -1 \text{ and } \omega^3 = 1.$$

Thus, (i) Left side $= \omega^3 a^2 + (\omega^2 + \omega^4) \, ab + \omega^3 b^2$
$$= a^2 - ab + b^2.$$

(ii) Left side $= a^2 - (\omega + \omega^2) \, ab + \omega^3 b^2$
$$= a^2 + ab + b^2.$$

(iii) Let side $= a^2 + b^2 \omega^3 + c^2 \omega^3 + ab \, (\omega + \omega^2)$
$$+ bc \, (\omega^2 + \omega^4) + ca \, (\omega + \omega^2)$$
$$= a^2 + b^2 + c^2 - bc - ca - ab.$$

Ex. 4. *Resolve into linear factors*

(i) $a^2 - ab + b^2$.   (ii) $a^3 + b^3$.

(iii) $a^2 + b^2 + c^2 - bc - ca - ab$.   [*C. U. 1930*]

(iv) $a^3 + b^3 + c^3 - 3abc$.

(i) $a^2 - ab + b^2 = a^2 + (\omega + \omega^2)\, ab + b^2 \omega^3$
$$= (a + \omega b)(a + \omega^2 b).$$

(ii) $a^3 + b^3 = (a + b)(a^2 - ab + b^2) = (a + b)(a + \omega b)(a + \omega^2 b).$

(iii) Given exp. $= a^2 + a^2 \omega^3 + c^2 \omega^3 + ab\,(\omega + \omega^2)$
$$+ bc\,(\omega^4 + \omega^2) + ca\,(\omega + \omega^2)$$
$$= a\,(a + \omega^2 b + \omega c) + \omega b\,(a + \omega^2 b + \omega c)$$
$$+ \omega^2 c\,(a + \omega^2 b + \omega c)$$
$$= (a + \omega^2 b + \omega c)(a + \omega b + \omega^2 c).$$

(iv) Given exp. $= (a + b + c)(a^2 + b^2 + c^2 - bc - ca - ab)$
$$= (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c).$$

Ex. 5. *Express $(a^2 + b^2)(c^2 + d^2)(e^2 + f^2)$ as the sum of two squares.*

$(a^2 + b^2)(c^2 + d^2) = (a + ib)(a - ib)(c + id)(c - id)$
$$= \{(a + ib)(c + id)\}\{(a - ib)(c - id)\}$$
$$= \{(ac - bd) + i\,(ad + bc)\}\{(ac - bd) - i\,(ad + bc)\}$$
$$= (ac - bd)^2 + (ad + bc)^2 = A^2 + B^2 \text{ say.}$$

$\therefore$ Given exp. $= (A^2 + B^2)(e^2 + f^2)$
$$= (Ae - Bf)^2 + (Af + Be)^2, \text{ as before}$$
$$= \{(ac - bd)\,e - (ad + bc)\,f\}^2 + \{(ac - bd)\,f + (ad + bc)\,e\}^2.$$

Note. In this way, the product of any number of factors each of which is the sum of two squares, can be expressed as the sum of two squares.

Ex. 6. *Show that $(1 + \omega - \omega^2)^3 - (1 - \omega + \omega^2)^3 = 0$.*

Since, $1 + \omega + \omega^2 = 0$, we get $1 + \omega = -\omega^2$ and $1 + \omega^2 = -\omega$.

$\therefore$ Left side of the given expression $= (-2\omega^2)^3 - (-2\omega)^3$
$$= -8\omega^6 - (-8\omega^3)$$
$$= -8 + 8 = 0.$$

Ex. 7. *If $\sqrt{a + ib} = x + iy$, show that $\sqrt{a - ib} = x - iy$.*

Since, $\sqrt{a + ib} = x + iy$, therefore squaring both sides

$a + ib = x^2 - y^2 + 2ixy$;

$\therefore \qquad a = x^2 - y^2, \; b = 2xy$;

$\therefore \quad a - ib = x^2 - y^2 - 2ixy = x^2 + (iy)^2 - 2ixy = (x - iy)^2$;

$\therefore \quad \sqrt{a - ib} = x - iy.$

Examples V

1. Find the square roots of

(i) $8 + 6i$.   (ii) $-1 + 2\sqrt{-2}$.   (iii) $21 - 20i$.

(iv) $2i$.   (v) $-3 + 4\sqrt{-7}$.   (vi) $a^2 - 1 + 2ia$.

(vii) $a + i\sqrt{1 - a^2}$.   (viii) $x + i\sqrt{x^4 + x^2 + 1}$.

(ix) $\dfrac{2 - 36i}{2 + 3i}$.   (x) $x^2 + \dfrac{1}{x^2} - \dfrac{4}{i}\left(x + \dfrac{1}{x}\right) - 2$.

2. Show that

(i) $\dfrac{(1 + i)^2 + (1 - i)^2}{(1 + i)^2 - (1 - i)^2} = 0$.

(ii) $(1 - i)^{-2} - (1 + i)^{-2} = i$.

(iii) $\dfrac{1 + 2i + 3i^2}{1 - 2i + 3i^2} = -i$.

(iv) $(3 + 4i)^{-\frac{1}{2}} + (3 - 4i)^{-\frac{1}{2}} = \frac{4}{5}$.

(v) $\sqrt{(4 + 3i\sqrt{20})} + \sqrt{(4 - 3i\sqrt{20})} = 6$.

3. Prove that

(i) $\sqrt{i} + \sqrt{-i} = \sqrt{2}$.   (ii) $\{\frac{1}{2}(1 + \sqrt{-3})\}^6 = 1$.

(iii) $\{\frac{1}{2}(-1 + \sqrt{-3})\}^{21} + \{\frac{1}{2}(-1 - \sqrt{-3})^{21} = 2$.

4. Express in the form $A + iB$

(i) $\dfrac{a + ib}{c + id}$.   (ii) $\dfrac{(1 + i)^2}{1 - i}$.

(iii) $\dfrac{x + iy}{x - iy} - \dfrac{x - iy}{x + iy}$.   (iv) $\dfrac{1 + i}{1 - i}$.

(v) $\dfrac{\sqrt{3} - i\sqrt{2}}{2\sqrt{3} - i\sqrt{2}}$.   (vi) $(1 + i)(1 + 2i)(1 + 3i)$.

(vii) $\dfrac{a + bi + ci^2 + di^3}{a - bi + ci^2 - di^3}$.

5. Find the moduli of the following expressions

(i) $\dfrac{3+4i}{12+5i}$.

(ii) $\dfrac{(3+i)(1+2i)}{(2-i)(2+3i)}$.

6. Simplify

(i) $\dfrac{5+4i}{4-3i}+\dfrac{5-4i}{4+3i}$.

(ii) $\dfrac{a+ib}{a-ib}+\dfrac{a-ib}{a+ib}$.

(iii) $\dfrac{\sqrt{1+x}+i\sqrt{1-x}}{\sqrt{1+x}-i\sqrt{1-x}}$, where $x=\dfrac{2a}{a^2+1}$.

(iv) $\dfrac{(x+i)^3-(x-i)^3}{(x+i)^2-(x-i)^2}$.

(v) $\left(\dfrac{4+3i}{3-2i}\right)^2-\left(\dfrac{4-3i}{3+2i}\right)^2$.

7. (i) If $x=3+2i$, show that
$$x^4-4x^3+4x^2+8x+39=0.$$

(ii) If $x=3+2i$, $y=3-2i$, show that
$$x^2+xy+y^2=23.$$

8. Show that a real value of x will satisfy the equation
$$\dfrac{1-ix}{1+ix}=a-ib,\text{ if }a^2+b^2=1. \qquad [\ C.\ U.\ 1933\ ]$$

9. (i) If $(a+ib)(c+id)=x+iy$, then
$$(a-ib)(c-id)=x-iy.$$

(ii) $\dfrac{a+ib}{c+id}=x+iy$, then $\dfrac{a-ib}{c-id}=x-iy$.

10. (i) If $\sqrt[3]{a+ib}=x+iy$, then $\sqrt[3]{a-ib}=x-iy$.

(ii) If $\sqrt[3]{a+ib}=x+iy$, then show that
$$4(x^2-y^2)=\dfrac{a}{x}+\dfrac{b}{y}.$$

11. Find the values of : (a) $(1+i)^{\frac{1}{2}}$. (b) $(-7+24i)^{\frac{1}{4}}$.

12. İf $a = \frac{1}{2}(-1 + \sqrt{-3})$ and $\beta = \frac{1}{2}(-1 - \sqrt{-3})$,

     show that $a^4 + a^2\beta^2 + \beta^4 = 0$.

13.   Prove that

   (i) $(1 - \omega + \omega^2)(1 + \omega - \omega^2) = 4$.

   (ii) $(1 - \omega + \omega^2)^4 + (1 + \omega - \omega^2)^4 = -16$.

   (iii) $(1 - \omega)(1 - \omega^2)(1 - \omega^4)(1 - \omega^8) = 9$.

   (iv) $\dfrac{x + \omega y + \omega^2 z}{y + \omega z + \omega^2 x} = \omega$.

14.   Prove that $(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3$

           $= (2a - b - c)(2b - c - a)(2c - a - b)$,

and $= 27abc$, if $a + b + c = 0$.

15.   If $(1 + x)^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_n x^n$, show that

     $(a_0 - a_2 + a_4 - \cdots\cdots)^2 + (a_1 - a_3 + a_5 - \cdots\cdots)^2$

       $= a_0 + a_1 + a_2 + \cdots\cdots + a_n.$

16.   If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_{2n} x^{2n}$,

     then $a_0 + a_3 + a_6 + \cdots\cdots = 3^{n-1}$.

17.   Prove that

$$\left\{\frac{-1 + \sqrt{(-3)}}{2}\right\}^n + \left\{\frac{-1 - \sqrt{(-3)}}{2}\right\}^n = 2, \quad \text{if } n \text{ be}$$

a multiple of 3, and $= -1$, if n be any other integer.

18. (i) Show that

     $(1 + x^2)(1 + y^2)(1 + z^2)$

         $= (1 - yz - zx - xy)^2 + (x + y + z - xyz)^2$.

   (ii) If $x = a + b$, $y = a + b\omega$, $z = a + b\omega^2$, show that

     $x^3 + y^3 + z^3 = 3(a^3 + b^3)$.

19. Show that

(i) $(x + y\omega + z\omega^2)^2 + (x\omega + y\omega^2 + z)^2$
$$+ (x\omega^2 + y + z\omega)^2 = 0.$$

(ii) $(1 - \omega + \omega^2)(1 - \omega^2 + \omega^4)(1 - \omega^4 + \omega^8)\cdots\cdots$
$$\text{to } 2n \text{ factors} = 2^{2n}.$$

20. If $x : y = a + ib : c + id$, show that
$$(c^2 + d^2) x^2 - 2(ac + bd) xy + (a^2 + b^2) y^2 = 0.$$

21. Show that
$$\sqrt{[-1 - \sqrt{\{-1 - \sqrt{(-1 - \cdots\cdots \text{ to } \infty)}\}}]} = \omega \text{ or, } \omega^2.$$

ANSWERS

1. (i) $\pm(3 + i)$.　　　(ii) $\pm(\sqrt{-2} + 1)$.　　　(iii) $\pm(5 - 2i)$.

(iv) $\pm(1 + i)$.　　　(v) $\pm\{2 + \sqrt{(-7)}\}$.　　　(vi) $\pm(a + i)$.

(vii) $\pm\dfrac{1}{\sqrt{2}}(\sqrt{1 + a} + i\sqrt{1 - a})$.

(viii) $\pm\dfrac{1}{\sqrt{2}}\{\sqrt{x^2 + x + 1} + i\sqrt{x^2 - x + 1}\}$.

(ix) $\pm(3i - 1)$.　　　　　　　(x) $\pm(x + x^{-1} + 2i)$.

4. (i) $\dfrac{ac + bd}{c^2 + d^2} + i\dfrac{bc - ad}{c^2 + d^2}$.　　　(ii) $-1 + i$.

(iii) $0 + i\dfrac{4xy}{x^2 + y^2}$.　　　　　(vi) $0 + i$.　　(v) $\dfrac{4}{7} - i\dfrac{\sqrt{6}}{14}$.

(vi) $-10 + i0$.　　　　　　(vii) $\dfrac{(a - c)^2 - (b - d)^2 + 2i(a - c)(b - d)}{(a - c)^2 + (b - d)^2}$.

5. (i) $\tfrac{5}{13}$.　　　(ii) $\sqrt{(\tfrac{10}{13})}$.　　　**6.** (i) $\tfrac{16}{25}$.

(ii) $\dfrac{2(a^2 - b)}{a^2 + b^2}$.　　　　　(iii) $\dfrac{2a}{a^2 + 1} + i\dfrac{a^2 - 1}{a^2 + 1}$.

(iv) $\dfrac{3x^2 - 1}{2x}$.　　　　　(v) $\dfrac{408}{169}i$.

11. (a) $\pm\dfrac{1}{\sqrt{2}}\left\{(\sqrt{2} + 1)^{\frac{1}{2}} + i(\sqrt{2} - 1)^{\frac{1}{2}}\right\}$.　　　(b) $\pm(2 + i)$,　$\pm i(2 + i)$.

SUPPLIMENT TO CHAPTER V

GEOMETRICAL REPRESENTATION OF A COMPLEX NUMBER

32·1. Geometrical interpretation of the imaginary number $i = \sqrt{-1}$.
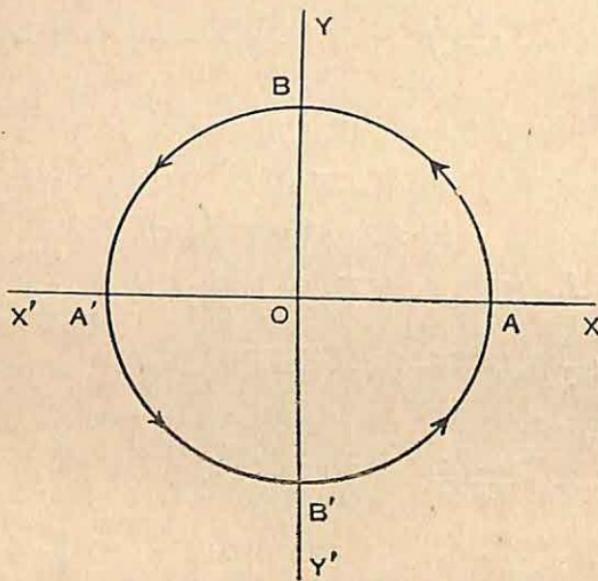
FIG. 1

Let $X'OX$ and $Y'OY$ be two straight lines cutting each other at right angles at O. O is called the origin, $X'OX$ is called the x-axis and $Y'OY$ the y-axis.

With centre O and radius $= 1$, we draw a circle cutting the x-axis at A and A', and the y-axis at B and B'.

If we now represent real numbers by points on the y-axis, (called the *real* axis), then A shall denote the

point $+1$ (for $OA = 1$, and A is on the positive side of the x-axis), and A' shall denote the point -1, with an explanation similar to that for A.

Now, $1 \times (-1) = -1$, and $\angle AOA' = 2$ right angles. Thus, multiplication by -1 may be regarded as an *operation* which turns the given length OA through two right angles, in the anti-clockwise sense, the magnitude remaining unaltered.

Again, by definition, $\sqrt{-1} \times \sqrt{-1} = -1$. Hence, $\sqrt{-1}$ may be regarded as an operator which rotates a given length OA through *one* right angle in the positive sense, and brings the point A to the point B on the y-axis.

In our figure, therefore, we may denote the point B by the number $+\sqrt{-1}$.

Again, since, $(\sqrt{-1})^3 = -\sqrt{-1}$, the operator $(\sqrt{-1})^3$, multiplying OA, shall bring the point A to the position B' on OY, after rotation through three right angles. Thus B' may be taken to represent the number $-\sqrt{-1}$.

Thus, we may introduce the convention that points on $Y'OY$, all represent purely imaginary numbers, and the y-axis may be called the *imaginary* axis.

32·2. Geometrical representation of Complex Numbers.

In order to represent any real number x geometrically, a straight line $X'OX$ (called the *real* axis) is taken with a fixed point O on it chosen as the origin to represent the number *zero*.

If A and L be points on $X'OX$, so that $OA = 1$, $(OL/OA) = x$, then the point L may be taken to represent

the real number x. If x be negative, the point L should be on OX'.
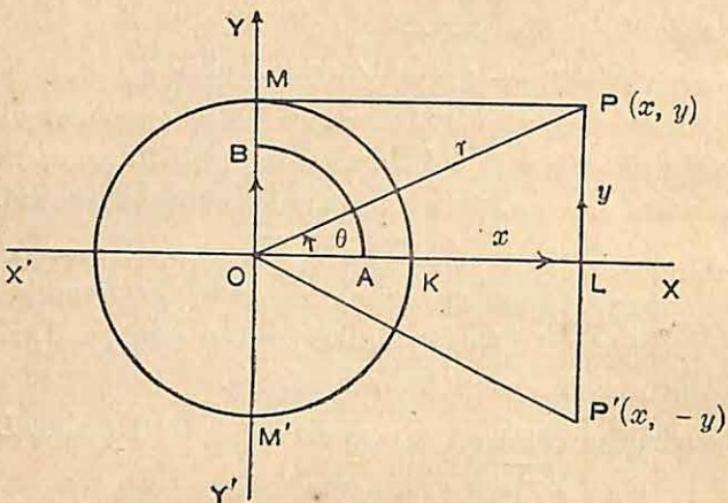
FIG. 2

For geometrical representation of a purely imaginary number $y\sqrt{-1}$, where y is real, we should, as in § 32·1 above, draw a straight line $Y'OY$ perpendicular to $X'OX$, ($Y'OY$ may be called the y-axis or the *imaginary* axis, as before), and on it take the points B and M, so that $OB = OA = 1$ in length and $(OM/OB) = y$. Then the point M may be taken to represent the purely imaginary number $y\sqrt{-1}$. Similarly, the point M' (where $OM' = OM$ in length) on OY' shall represent $-y\sqrt{-1}$.

Finally, for the geometrical representation of complex numbers of the type $x + iy$, where x and y are two real numbers and $i = \sqrt{-1}$, we naturally require a *two*-dimensional space, (for simplicity, **a plane** on which $X'OX$ and $Y'OY$ are the rectangular Cartesian axes of co-ordinates). Thus, a complex number $z \equiv x + iy$ being given if a point P,

(as in fig. 2), on the above plane be taken whose Cartesian co-ordinates are (x, y), and polar co-ordinates (r, θ), then P corresponds *uniquely* to the number z, and is called the point z which it geometrically represents.

Note 1. Students, who are familiar with vector notation used in Statics and Dynamics, may observe that $\vec{z} = \overrightarrow{OP} = \overrightarrow{OL} + \overrightarrow{LP} = \overrightarrow{OL} + \overrightarrow{OM}$ $= \vec{x} + \vec{iy}$, where $\vec{x} = \overrightarrow{OL}$ and $\vec{iy} = \overrightarrow{OM}$; PL is here perpendicular to OX and PM to OY.

Note 2. It is to be noted that corresponding to every given complex quantity $z \equiv x + iy$, there is a *unique* position of the point $P(x, y)$ on the xy-plane; and *conversely*, every point $P(x, y)$ on the xy-plane represents a complex number $z \equiv x + iy$. This is known as the principle of **one-to-one correspondence**.

Def. The figure 2, above, containing the real and the imaginary axes, in the plane of which complex numbers, such as 'z', are geometrically represented, is called the **Argand Diagram**, and the plane is often spoken of as the **z-plane**.

32·3. The Conjugate Complex Number.

Def. The complex quantity $(x - iy)$ is said to be **conjugate** to the complex quantity $(x + iy)$; and *vice versa*.

If in fig. 2, PL be produced to P', so that $LP' = PL$, then z' the complex number corresponding to P' is $\vec{z'} = \overrightarrow{OP'}$ $= \overrightarrow{OL} + \overrightarrow{LP'} = \vec{x} - \overrightarrow{LP} = \vec{x} - \vec{iy}$; *i.e.*, $z' = x - iy$, which is the conjugate to $z \equiv x + iy$. Thus, the conjugate of a complex number is its image in the real axis.

Note. If $z \equiv x + iy$ and $z' \equiv x - iy$, then prove geometrically that $z + z' = 2x$, a real quantity. [This at once follows from fig. 2 by using vector notation.]

32·4. The Modulus and the Amplitude of z ≡ x + iy.

In fig. 2, $r =$ the positive value of the radius vector OP, where the point P represents $z \equiv x + iy$; and $\theta =$ the vectorial angle XOP, measured positive *anti-clockwise*.

Then, $r = + \sqrt{x^2 + y^2}$ is called the **Modulus** of z and is written as *mod. z* or as $|z|$. Also, the angle θ is called the **Amplitude** or the **Argument** of z and is written as *amp. z*.

Note. Since $x = r \cos \theta$, $y = r \sin \theta$, as is evident from the figure, we evidently have, $z \equiv x + iy = r (\cos \theta + i \sin \theta)$.

32·5. Addition of complex Quantities.
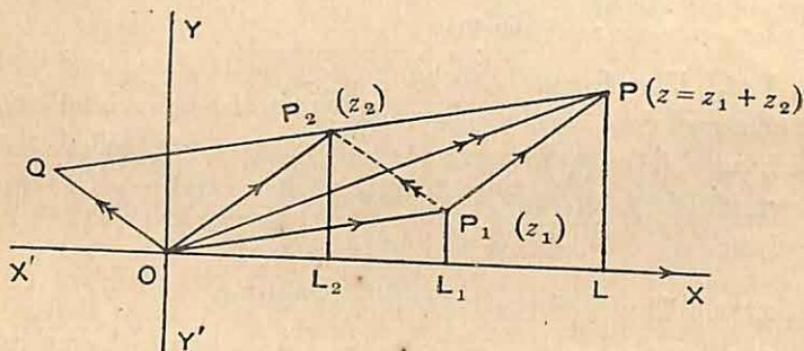
FIG. 3

Let P_1 and P_2 represent the complex numbers, $z_1 \equiv x_1 + iy_1$, and $z_2 \equiv x_2 + iy_2$, respectively. Complete the parallelogram, OP_1PP_2. Then, $z = \overrightarrow{OP} = \overrightarrow{OP_1} + \overrightarrow{P_1P} = \overrightarrow{OP_1} + \overrightarrow{OP_2}$, ($\because OP_2$ is equal and parallel to P_1P). $= \overrightarrow{z_1} + \overrightarrow{z_2}$.

$\therefore P$ represents the sum of the two given complex numbers, z_1 and z_2, *i.e.*, the complex number, $z_1 + z_2 = (x_1 + iy_1) + (x_2 + iy_2) = (x_1 + x_2) + i(y_1 + y_2)$.

Note. From fig. 3, $OP < OP_1 + P_1P = OP_1 + OP_2$, in magnitude.

$\therefore \quad |z_1 + z_2| < |z_1| + |z_2|$,

or, $\quad \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2} < \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2}$.

The above inequality becomes an equality, when O, P_1, P (and therefore, O, P_1, P_2) are collinear, in which case, $y_1/x_1 = y_2/x_2$, or, $(x_1 y_2 - x_2 y_1) = 0$.

Ex. 2. If z_1, z_2,..., z_n be n complex numbers, prove that $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$.

32·6. Subtraction of Complex Numbers.

In fig. 3, if we join the diagonal $P_1 P_2$, $\quad \vec{z_2} - \vec{z_1}$
$= \vec{OP_2} - \vec{OP_1} = \vec{OP_2} + \vec{P_1O} = \vec{P_1O} + \vec{OP_2} = \vec{P_1P_2} = \vec{OQ}$,
where $OP_1 P_2 Q$ is a parallelogram, (and from geometrical considerations, $P P_2 Q$ is a straight line).

Note. From $\triangle OP_1P_2$, we have, $P_1P_2 > OP_1 \sim OP_2$, in magnitude, (unless $x_1 y_2 = x_2 y_1$, as before).

$\therefore \quad |z_1 - z_2| > |z_1| \sim |z_2|$.

32·7. Multiplication of Complex Numbers.
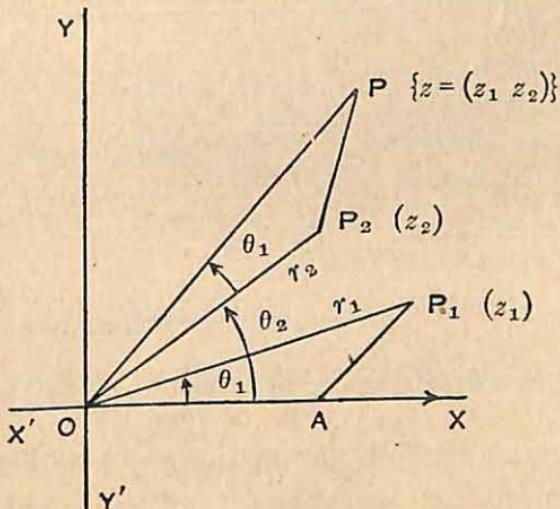
FIG. 4

Let $z_1 \equiv r_1 (\cos \theta_1 + i \sin \theta_1)$ and $z_2 \equiv r_2 (\cos \theta_2 + i \sin \theta_2)$ be two complex numbers expressed in terms of their moduli and amplitudes. Then by actual multiplication, $z_1 z_2 = r_1 r_2 \{(\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2)$

$$+ i (\sin \theta_1 \cos \theta_2 + \cos \theta_1 \sin \theta_2)\}, \; [\; \because \; i^2 = -1\;],$$
$$= r_1 r_2 \{\cos (\theta_1 + \theta_2) + i \sin (\theta_1 + \theta_2)\}.$$

Hence, $|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|$, and

amp. $(z_1 z_2) = \theta_1 + \theta_2 = $ amp. $z_1 + $ amp. z_2.

Note. In fig. 4, $OA = 1, OP_1 = r_1, OP_2 = r_2, OP = r_1 r_2, \angle XOP_1 = \theta_1,$ $\angle XOP_2 = \theta_2, \angle XOP = \theta_1 + \theta_2$. $\therefore \;\; \angle P_2 OP = \theta_1$.

Thus, $OA/OP_1 = 1/r_1 = r_2/r_1 r_2 = OP_2/OP,$

and $\angle AOP_1 = \theta_1 = \angle P_2 OP$. $\therefore \;\; \triangle AOP_1$ and $\triangle P_2 OP$ are similar.

Ex. 3. If $z \equiv x + iy$ and $z' \equiv x - iy$, prove that $zz' = x^2 + y^2$, a real number.

32·8. Division of one complex number by another.

Let $z_1 \equiv r_1 (\cos \theta_1 + i \sin \theta_1), z_2 \equiv r_2 (\cos \theta_2 + i \sin \theta_2)$ be the two complex quantities. Then,

$$\frac{z_1}{z_2} = \left(\frac{r_1}{r_2}\right) \cdot \left\{ \frac{(\cos \theta_1 + i \sin \theta_1)(\cos \theta_2 - i \sin \theta_2)}{(\cos^2 \theta_2 - i^2 \sin^2 \theta_2)} \right\} \quad \cdots \quad (1)$$

$$= \frac{r_1}{r_2} \left\{ \cos (\theta_1 - \theta_2) + i \sin (\theta_1 - \theta_2) \right\}.$$

Since the denominator of the second factor of (1) is $\cos^2 \theta_2 + \sin^2 \theta_2 = 1.$

$$\therefore \qquad \left| \frac{z_1}{z_2} \right| = \frac{r_1}{r_2} = |z_1| / |z_2|,$$

and amp. $\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = $ amp. $z_1 - $ amp. z_2.

Note. The **reciprocal** of a complex number $z \equiv (x + iy)$ is $\dfrac{1}{z}$, where $\left| \dfrac{1}{z} \right| = \dfrac{1}{\sqrt{x^2 + y^2}}$, and amp. $\left(\dfrac{1}{z} \right) = -$ amp. z, as the amplitude of the real number $1 \equiv (\cos 0 + i \sin 0)$ is *zero*.

QUADRATIC EQUATIONS

33. Definition.

If the highest power of the unknown quantity in an equation be two, the equation is called an *equation of the second degree* or a **quadratic equation** or simply a *quadratic* in the unknown quantity.

Thus, $x^2 - 6 = 3$, $5x^2 - 6x + 2 = 0$ are quadratic equations in x.

The *general form* of a quadratic equation in x is $ax^2 + bx + c = 0$, where $a \,(\neq 0)$, b, c, are any quantities whatever. The term independent of x i.e., c is called the *constant term* or the *absolute term*.

Quadratic equations are either pure or adfected. A quadratic equation which contains only the second power of the unknown quantity and not the first, is called a *Pure Quadratic* and a quadratic which contains the second as well as the first power of the unknown quantity, is called an *Adfected Quadratic*.

Thus, $3x^2 = 27$ is a pure quadratic and $2x^2 - 15x + 7 = 0$ is an adfected quadratic.

Note. For sake of brevity, any expression involving x, is sometimes denoted by the symbolic notation $f(x)$, $F(x)$, $\phi(x)$, $\psi(x)$ etc. Thus $f(x) = ax^2 + bx + c$; $\psi(x) = x^2 - 5$.

By $f(a)$ we would mean the value that we shall get by putting a for x in $ax^2 + bx + c$; i.e., $f(a) = aa^2 + ba + c$; $\phi(2) = 2^2 - 5 = -1$; $f(o) = c$; $\phi(o) = -5$ etc.

34. Solution of Pure Quadratics.

Every pure quadratic equation can be reduced to the form
$$x^2 = a.$$

By extracting the square root of both sides of this equation, we get $\pm x = \pm \sqrt{a}$, which are obviously equivalent to
$$x = \pm \sqrt{a}.$$

Note 1. Thus, it is clear that in extracting the square roots of both sides of an equation, it will be sufficient to attach the double sign to one side only.'

Note 2. Every pure quadratic has two and only two roots, which are equal in magnitude but opposite in sign.

35.   Solution of Adfected Quadratics.

There are two principal methods for solving adfected quadratics ;

(i) *Method of factorisation.*

(ii) *Method of completing the square.*

The first method, that is, the method of factorisation is applicable in cases when the factors of the quadratic expression can easily be determined. If the factors cannot be so determined, the other method, that is, the method of completing the square is used.

35(a).   The method of factorisation.

The method is best illustrated by the solution of the following numerical example, *viz.*, $4x^2 - 16x + 15 = 0$.

Here the factors of the quadratic expression $4x^2 - 16x + 15$ can easily be determined.   Thus,

$$4x^2 - 16x + 15 = 0, \ \ or, \ \ 2x\,(2x - 3) - 5\,(2x - 3) = 0,$$

$$i.e., \ \ (2x - 5)(2x - 3) = 0.$$

$\therefore$ Either, $2x - 3 = 0$,   whence $x = \frac{3}{2}$,

or, $2x - 5 = 0$,   whence $x = \frac{5}{2}$.

Note.   This can also be done by the second method.

35(b).   The method of completing the square.

Let the quadratic equation be of the form $ax^2 + bx + c = 0$.

Dividing by a and transposing, we get

$$x^2 + \frac{b}{a}\,x = -\,\frac{c}{a}\,,$$

$$\text{or,} \quad x^2 + \frac{b}{a}\,x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a},$$

$$\text{or,} \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

Extracting the square root,

$$x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a},$$

$$\text{or,} \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$i.e., \quad \mathbf{x} = \frac{\mathbf{b} - \pm\sqrt{\mathbf{b^2 - 4ac}}}{\mathbf{2a}}.$$

It is thus clear that the standard quadratic equation $ax^2 + bx + c = 0$ has two roots, namely,

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Note 1. The famous Hindu Mathematician Sridhar Acharya has given the following method of completing the square, thereby obtaining the roots of a quadratic.

Multiplying both sides of $ax^2 + bx + c = 0$ by $4a$ and transposing,

$$\text{we get} \quad 4a^2x^2 + 4abx = -4ac,$$

$$\text{or,} \quad 4a^2x^2 + 4abx + b^2 = b^2 - 4ac,$$

$$\text{or,} \quad (2ax + b)^2 = b^2 - 4ac,$$

$$\text{or,} \quad 2ax + b = \pm\sqrt{b^2 - 4ac} ;$$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Cor. 1. When the equation is written in the form $ax^2 + 2bx + c = 0$, the roots are $\dfrac{-b + \sqrt{b^2 - ac}}{a}$ and $\dfrac{-b - \sqrt{b^2 - ac}}{a}$.

Cor. 2. If the equation is of the form $x^2 + px + q = 0$, the roots are $\dfrac{-p + \sqrt{p^2 - 4q}}{2}$ and $\dfrac{-p - \sqrt{p^2 - 4q}}{2}$.

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Note 2. In solving a given quadratic equation with numerical coefficients, the simple method will be to make use of the above formula by substituting the values of a, b and c.

Thus, to solve the equation $5x^2 - 2x - 3 = 0$, we have $a = 5$, $b = -2$ and $c = -3$; hence by the above formula, we get,

$$= \frac{2 \pm \sqrt{4 - 4 \times 5 \times (-3)}}{2.5} = \frac{2 \pm \sqrt{4 + 60}}{10}$$

$$= \frac{2 \pm 8}{10} = 1, \quad \text{or,} \quad -\frac{3}{5}.$$

35(c).   Particular cases of the general quadratic.

(1) When the constant term $c = 0$, the equation reduces to $ax^2 + bx = 0$, or, $x(ax + b) = 0$ and the roots are

$$\frac{-b \pm \sqrt{b^2 - 0}}{2a}, \text{ i.e., } 0 \text{ and } -\frac{b}{a}.$$

(2) When $b = 0$, the equation reduces to the pure quadratic $ax^2 + c = 0$ whose roots are

$$\frac{-0 \pm \sqrt{0 - 4ac}}{2a} = \pm \sqrt{\frac{-c}{a}}.$$

(3) When $b = 0$, $c = 0$, the equation reduces to $ax^2 = 0$, which has got two zero roots.

35(d).   Special artifices of solution.

In some cases, complicated equations may be easily solved by applying special artifices. The following examples will serve as illustrations :

Ex. 1.   *Solve* $(x - 1)(x - 2) = \dfrac{15}{14^2}$.

Let $a = 14$. Then the equation becomes $(x - 1)(x - 2) = \dfrac{a + 1}{a^2}$,

or, $a^2 (x - 1)(x - 2) - a - 1 = 0$,

i.e., $(ax - a)(ax - 2a) + (ax - 2a) - (ax - a) - 1 = 0$,

or, $(ax - 2a)(ax - a + 1) - (ax - a + 1) = 0$,

i.e., $(ax - a + 1)(ax - 2a - 1) = 0$,

whence $ax - a + 1 = 0$, $ax - 2a - 1 = 0$,

when $ax - a + 1 = 0$, $x = \dfrac{a-1}{a} = \dfrac{13}{14}$,

when $ax - 2a - 1 = 0$, $x = \dfrac{2a+1}{a} = \dfrac{29}{14} = 2\dfrac{1}{14}$.

Ex. 2. *Solve* $\dfrac{x-a}{x-b} + \dfrac{x-b}{x-a} = \dfrac{a}{b} + \dfrac{b}{a}$.

Here $x = 0$ is evidently a solution, for transposing, we have

$$\left(\dfrac{x-a}{x-b} - \dfrac{a}{b}\right) + \left(\dfrac{x-b}{x-a} - \dfrac{b}{a}\right) = 0,$$

or, $\qquad \dfrac{(b-a)\,x}{b\,(x-b)} + \dfrac{(a-b)\,x}{a\,(x-a)} = 0,$

i.e., $\qquad (a-b)x\left\{\dfrac{1}{a\,(x-a)} - \dfrac{1}{b\,(x-b)}\right\} = 0,$

whence $(a-b)\,x = 0$, *i.e.*, $x = 0$.

The other root is obtained from $\left\{\dfrac{1}{a\,(x-a)} - \dfrac{1}{b\,(x-b)}\right\} = 0,$

or, $\dfrac{1}{a\,(x-a)} = \dfrac{1}{b\,(x-b)}$, whence $x = \dfrac{a^2 - b^2}{a - b} = a + b.$

36. Equations reducible to Quadratics.

There are various types of equations, which though not quadratic in form can, by a suitable transformation or a simple artifice, be reduced to quadratic forms. This is illustrated by the following examples :

Ex. 1. *Solve* $4\sqrt{(x-1)} - \sqrt{(x+4)} = \sqrt{(x+20)}$.

Squaring both sides, $16(x-1) + (x+4) - 8\sqrt{(x^2 + 3x - 4)} = x + 20$.

Transposing and reducing, $2x - 4 = \sqrt{(x^2 + 3x - 4)}$.

Squaring again, $\quad 4x^2 - 16x + 16 = x^2 + 3x - 4$,

or, $\quad 3x^4 - 19x + 20 = 0$, or, $(3x-4)(x-5) = 0$,

$\therefore \ x = \tfrac{4}{3}$, or, 5.

Now, the root 5 is found to satisfy the original equation but the other root $\tfrac{4}{3}$ does not satisfy it, as can be easily verified by substitution, when $\sqrt{(x+4)}$ is taken to represent the positive square root of $x + 4$.

Thus, $x = 5$ is a root of the given equation *but $x = \frac{4}{3}$ is not a root*.

In fact $\frac{4}{3}$ is the root of the equation

$$4\sqrt{(x-1)} + \sqrt{(x+4)} = \sqrt{(x+20)}.$$

Not. Extraneous Solutions. If the process of solving an equation involves the clearing of radical as in the above example, then the *values of x obtained are not all necessarily the solution of the original equation.* Hence in solving equations of this type, care should be taken to verify in each case, the roots thus obtained; and while verifying, it should be noted that the square roots (unless otherwise specified) are considered to be positive.

Ex. 1. *Solve $4x^2 + 6x + \sqrt{(2x^2 + 3x + 4)} = 13$.*

Adding 8 to both sides, we have

$$2(2x^2 + 3x + 4) + \sqrt{(2x^2 + 3x + 4)} = 21.$$

Putting $y = \sqrt{(2x^2 + 3x + 4)}$, the equation reduces to $2y^2 + y = 21$, whence we get easily $(y-3)(2y+7) = 0$, $\therefore y = 3$, or, $-\frac{7}{2}$.

Since y is not negative, we reject the value $-\frac{7}{2}$ of y.

When $y = 3$, $\sqrt{2x^2 + 3x + 4} = 3$.

$\therefore$ $2x^2 + 3x + 4 = 9$,   or, $2x^2 + 3x - 5 = 0$. $\therefore$ $(x-1)(2x+5) = 0$.

$\therefore$ $x = 1$,   or, $-\frac{5}{2}$.

Ex. 3. *Solve $\sqrt{(x^2 - 2x + 49)} - \sqrt{(x^2 - 2x + 16)} = 3$.* $\cdots$ (1)

We have *identically*,

$$(x^2 - 2x + 49) - (x^2 - 2x + 16) = 33.$$ $\cdots$ (2)

Dividing (2) by the given equation (1), we have

$$\sqrt{(x^2 - 2x + 49)} + \sqrt{(x^2 - 2x + 16)} = 11.$$ $\cdots$ (3)

Adding the equations (1) and (3), and dividing by 2, we have

$$\sqrt{(x^2 - 2x + 49)} = 7 ; \therefore x^2 - 2x + 49 = 49 ;$$

$\therefore$ $x^2 - 2x = 0$,   *i.e.*, $x(x-2) = 0$, $\therefore$ $x = 0$, or, 2.

Note. This above artifice is sometimes useful in solving irrational equations.

Ex. 4. *Solve $x^{\frac{2}{3}} - 5x^{\frac{1}{3}} + 6 = 0$.*

Putting $x^{\frac{1}{3}} = y$, the given equation reduces to

$$y^2 - 5y + 6 = 0, \text{ or, } (y-2)(y-3) = 0 ; \therefore y = 2, \text{ or, } 3.$$

$\therefore$ $x^{\frac{1}{3}} = 2$, or, 3, whence cubing, $x = 8$, or, 27.

Ex. 5. *Solve* $4^x - 3.2^{x+2} + 32 = 0$.

Since $4^x = (2^2)^x = (2^x)^2$, and $2^{x+2} = 2^x.2^2 = 4.2^x$, putting $2^x = y$, the given equation reduces to

$$y^2 - 12y + 32 = 0, \quad \text{or,} \quad (y-4)(y-8) = 0 ; \quad \therefore \ y = 4, \text{ or, } 8.$$

$\therefore \ 2^x = 4, \text{ or, } 8.$

When $2^x = 4 = 2^2$, $x = 2$; when $2^x = 8 = 2^3$, $x = 3$.

Ex. 6. *Solve* $(x+a)(x+2a)(x+3a)(x+4a) = a^4$.

Re-arranging the factors, $\{(x+a)(x+4a)\}\{(x+2a)(x+3a)\} = a^4$,

or, $(x^2 + 5ax + 4a^2)(x^2 + 5ax + 6a^2) = a^4$.

Putting $y = x^2 + 5ax$, we have $(y + 4a^2)(y + 6a^2) = a^4$,

or, $y^2 + 10a^2 y + 24a^4 = a^4$, or, $y^2 + 10a^2 y + 23a^4 = 0$;

$\therefore \ y = \dfrac{-10a^2 \pm a^2 \sqrt{100 - 92}}{2} = -5a^2 \pm a^2 \sqrt{2}$,

Hence, $x^2 + 5ax = -5a^2 \pm a^2 \sqrt{2}$, or, $x^2 + 5ax + (5a^2 \mp a^2 \sqrt{2}) = 0$.

$\therefore \ x = \dfrac{-5a \pm \sqrt{25a^2 - 4(5a^2 \mp a^2 \sqrt{2})}}{2}$

$\quad = \dfrac{a}{2}\left\{ -5 \pm \sqrt{5 \pm 4\sqrt{2}} \right\}.$

Examples VI

Solve the following equations :

1. $\dfrac{\sqrt{(x^2 + 4)} + \sqrt{(x+1)}}{\sqrt{(x^2 + 4)} - \sqrt{(x+1)}} = 3.$

[*Apply Componendo and Dividendo.*]

2. $\dfrac{x + \sqrt{(x^2 - 1)}}{x - \sqrt{(x^2 - 1)}} + \dfrac{x - \sqrt{(x^2 - 1)}}{x + \sqrt{(x^2 - 1)}} = 98.$

3. $(1+x)^{\frac{1}{3}} + (1-x)^{\frac{1}{3}} = 2^{\frac{1}{3}}.$ [*Cube both sides.*]

4. (i) $5^x + \dfrac{1}{5^x} = \dfrac{26}{5}.$ (ii) $x + \dfrac{2}{\sqrt{x}} = 3.$

5. $2\sqrt{(x+5)} - \sqrt{(2x+8)} = 2.$.   [*C. U. 1937*]

6. $\sqrt{2x-1} + \sqrt{3x-2} = \sqrt{4x-3} + \sqrt{5x-4}.$

7. $x^2 + x + 10\sqrt{(x^2 + 3x + 16)} = 2(20 - x).$

8. $\sqrt{(3x^2 - 2x + 9)} + \sqrt{(3x^2 - 2x - 4)} = 13.$

9. $\dfrac{2x + \sqrt{x}}{2x - \sqrt{x}} + 6\,\dfrac{2x - \sqrt{x}}{2x + \sqrt{x}} = 5.$

10. (i) $(x-1)(x-2)(x-3)(x-4) = 120.$
(ii) $(x+4)(x+6)(x-5)(x-7) = 504.$

11. $x^2 - 5x + 2\sqrt{(x^2 - 5x + 3)} = 12.$

12. $\sqrt{(6x^2 + 7x + 6)} - \sqrt{(7x^2 + 7x + 2)} = x + 2.$

13. $\sqrt{(x+2)} + \sqrt{(x-1)} = \sqrt{(2x-5)}.$

14. $x^{-1} + x^{-\frac{1}{2}} = 6.$ **15.** $4^{1+x} + 4^{1-x} = 10.$

16. $3^{2x} + 9 = 10.3^x.$ **17.** $\sqrt{(2^x)} + \dfrac{1}{\sqrt{(2^x)}} = 2.$

18. $2^{x+1} + 2^{2x} - 8 = 0.$ **19.** $2^{x^2} : 2^{2x} = 8 : 1.$

20. (i) $x(x-1)(x-2) = 9.8.7.$ (ii) $x^{\frac{1}{3}} + 9x^{-\frac{1}{3}} = 4.$

21. $\sqrt[3]{a-x} + \sqrt[3]{b-x} = \sqrt[3]{a+b-2x}.$

22. $\dfrac{x-3}{\sqrt{(x^2 - 6x + 36)}} = \dfrac{x-4}{\sqrt{(x^2 - 8x + 64)}}.$

23. $\sqrt{(55 - 21x + 2x^2)} - \sqrt{(30 - 11x + x^2)} = 5 - x.$

24. $\sqrt{(x+1)} - 4\sqrt{(x-4)} + 5\sqrt{(x-7)} = 0.$

25. $\sqrt{(x^2 - 3x + 2)} - \sqrt{(x^2 - 3x - 6)} = \sqrt{20} - \sqrt{12}.$

26. $\sqrt{(5 + \sqrt{x})} + \sqrt{(5 - \sqrt{x})} = \dfrac{12}{\sqrt{(5 + \sqrt{x})}}.$

27. $\dfrac{\sqrt{(x+1)} + \sqrt{(x-1)}}{\sqrt{(x+1)} - \sqrt{(x-1)}} + \dfrac{\sqrt{(x+1)} - \sqrt{(x-1)}}{\sqrt{(x+1)} + \sqrt{(x-1)}} = 2(\sqrt{x} + \tfrac{3}{4}).$

28. $\dfrac{a + 2x + \sqrt{a^2 + 2x^2}}{a + 2x - \sqrt{a^2 + 2x^2}} = \dfrac{2x}{a}.$

29. $\dfrac{1}{\sqrt{2+x} - \sqrt{2}} + \dfrac{1}{\sqrt{2-x} + \sqrt{2}} = \dfrac{\sqrt{2}}{x}.$

30. $(1+x)^{\frac{2}{3}} + 2(1-x)^{\frac{2}{3}} = 3(1-x^2)^{\frac{1}{3}}.$

31. $2\sqrt{(x^2-9x+18)} - \sqrt{(x^2-4x-12)} = x-6.$

32. $\dfrac{5}{x^2+6x+8} = \dfrac{1}{x^2+6x+5} + \dfrac{4}{x^2+6x+9}.$

33. $\sqrt{x^2-5x+4} + \sqrt{x^2-7x+12} = \sqrt{x^2-6x+8}.$

34. $\left(x-\dfrac{1}{x}\right)^2 + 2\left(x+\dfrac{1}{x}\right) = 7\tfrac{1}{4}.$

35. $9x^4-24x^3-2x^2-24x+9 = 0.$

36. $8x^4-42x^3+29x^2+42x+8 = 0.$

37. $(1+i)x^2 - (7+13i)x + 2 + 60i = 0,$ where $i=\sqrt{-1}.$

38. Find tne value of

$$[\sqrt{2}+\sqrt{\{2+\sqrt{(2+\cdots\cdots \text{ to infinity})\}}}].$$

39. The product of four consecutive numbers is 840 ; find them.

40. One-fifteenth of the product of the ages of a father and his son is equal to the age of the father increased by 40 years ; if the sum of their ages be 77 years, find the age of each.

ANSWERS

1. 4, 0. **2.** ± 5. **3.** ± 1. **4.** (i) ± 1. (ii) 1.

5. ± 4. **6.** 1. **7.** $0, -3$. **8.** $4, -\tfrac{10}{3}$.

9. $1, \tfrac{9}{4}$. **10.** (i) $6, -1, \tfrac{1}{2}(5\pm\sqrt{-39})$. (ii) $3, -2, 8, -7$.

11. $6, -1$. **12.** $-\tfrac{1}{2}, -\tfrac{2}{3}$. **13.** 2. **14.** $\tfrac{1}{4}$.

15. $\pm\tfrac{1}{2}$. **16.** 2, 0. **17.** 0. **18.** 1.

19. $3, -1$. **20.** (i) $9, -3\pm\sqrt{-47}$. (ii) 1, 27.

21. $a, b, \tfrac{1}{2}(a+b)$. **22.** $0, \tfrac{24}{7}$. **23.** 5. **24.** 8.

25. $6, -3$. **26.** 16, 9. **27.** $\tfrac{9}{4}$. **28.** $-\tfrac{1}{2}a, 2a$.

29. $-\sqrt{3}$. **30.** $0, \tfrac{7}{9}$. **31.** 6, 7. **32.** $1, -7$. **33.** 4.

34. $2, \tfrac{1}{2}, \tfrac{1}{4}(-9\pm\sqrt{65})$. **35.** $3, \tfrac{1}{3}, \tfrac{1}{3}(-1\pm\sqrt{-8})$. **36.** $4, -\tfrac{1}{4}, 2, -\tfrac{1}{2}$.

37. $7-2i, 3+5i$. **38.** 2. **39.** 4, 5, 6, 7 ; $-7, -6, -5, -4$. **40.** 50, 27.

SIMULTANEOUS QUADRATIC EQUATIONS
SEC. A. TWO UNKNOWN QUANTITIES

37. One of the Equations Linear.

The general method of solving simultaneous equations involving two unknown quantities of which one is linear and the other quadratic, is to express one of the unknown quantities, say y, in terms of the other, say x, from the linear equation and to substitute this value of y in the other equation, whereby a quadratic in x will be obtained and this can be easily solved. In some cases, the solutions can be effected by using special artifices, as illustrated in the Examples 3 below.

Ex. 1. *Solve* $x^2 + y^2 = 25$ ...   ... (1)

$\qquad\qquad x + y = 7.$ ...   ... (2)

From (2), $y = 7 - x.$ ...   ... (3)

Substituting this value of y in (1), we get

$\qquad x^2 + (7 - x)^2 = 25,$ or, $x^2 - 7x + 12 = 0.$

$\qquad \therefore \ (x - 3)(x - 4) = 0,$ whence $x = 3,$ or, $4.$

Now, from (3), $y = 4,$ or, $3.$

Thus, the roots of the equation are $x = 3,$ $y = 4$;

$\qquad\qquad\qquad$ or, $x = 4,$ $y = 3.$

Note. In example of this type, while stating the results, the students should take special care *to combine properly the corresponding values of x and y.*

Ex. 2. *Solve* $x^2 + xy + y^2 + x + y = 5$ ... (1)

$$x + y = 2. \qquad \cdots \quad (2)$$

Equation (1) can be written as

$$(x + y)^2 + (x + y) - xy - 5 = 0. \qquad \cdots \quad (3)$$

∴ Substituting the value of $y = 2 - x$ from (2) in (3), we get

$$2^2 + 2 - x (2 - x) - 5 = 0,$$

or, $\qquad x^2 - 2x + 1 = 0$, or, $(x - 1)^2 = 0$,

whence, $\quad x = 1, 1$ and $\quad ∴ \ y = 1, 1$ from (2).

Ex. 3. *Solve* $x + y = 8$ ... ... (1)

$$xy = 15. \qquad \cdots \qquad \cdots \quad (2)$$

We have, $(x - y)^2 = (x + y)^2 - 4xy = 64 - 60 = 4$.

$$∴ \ x - y = \pm 2. \qquad \cdots \qquad \cdots \quad (3)$$

Thus, we have, $x + y = 8$ ⎫
                                    ⎬ ... (4) or, $x + y = 8$ ⎫
                  $x - y = 2$ ⎭ $x - y = -2$ ⎬ ... (5)
                                                          ⎭

From (4), adding and subtracting, and dividing by 2, we get

$$x = 5, \quad y = 3.$$

Similarly from (5), we get $x = 3, y = 5$.

Thus, the roots of the given equations are

$$x = 5, y = 3 \ ; \quad \text{or}, \quad x = 3, y = 5.$$

Note. The example can also be solved by the method shown in Ex. 1.

Ex. 4. *Solve* $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{5}{2}$ ... (1)

$$x + y = 10 \qquad \cdots \quad (2)$$

[*C. U. 1938*]

Equation (1) can on simplification be written as

$$\frac{x + y}{\sqrt{xy}} = \frac{5}{2}, \qquad \text{or,} \qquad \frac{10}{\sqrt{xy}} = \frac{5}{2} \quad \text{from (2),}$$

or, $\quad \sqrt{xy} = 4$. $\qquad ∴ \ xy = 16.$ ... (3)

Substituting $y = 10 - x$ from (2) in (3), we get

$$x(10-x) = 16, \qquad \text{or,} \qquad x^2 - 10y + 16 = 0,$$

or, $(x-2)(x-8) = 0$, whence $x = 2$, or, 8 ;

and $\therefore$ $y = 3$, or, 2 from (2).

Thus, the roots of the equations are

$$x = 2, y = 8 ; \quad \text{or,} \quad x = 8, y = 2.$$

38. Miscellaneous Equations.

Ex. 5. *Solve* $x + \dfrac{4}{y} = 1$ $\qquad \cdots \qquad \cdots$ (1)

$$y + \dfrac{4}{x} = 25. \qquad \cdots \qquad \cdots \text{ (2)}$$

[*C. U. 1940*]

From (1), $\qquad xy + 4 = y.$ $\qquad \cdots \qquad \cdots$ (3)

From (2), $\qquad xy + 4 = 25x.$ $\qquad \cdots \qquad \cdots$ (4)

$\therefore$ From (3) and (4), $y = 25x.$ $\qquad \cdots \qquad \cdots$ (5)

Substituting this value of y in (1) and simplifying, we get

$$25x^2 - 25x + 4 = 0, \quad \text{or,} \quad (5x-1)(5x-4) = 0,$$

whence $x = \frac{1}{5}$, or $\frac{4}{5}$, and the corresponding values of y are 5, or, 20.

Thus, the roots are $x = \frac{1}{5}$, $y = 5$ and $x = \frac{4}{5}$, $y = 20$.

Ex. 6. *Slove* $x^3 + y^3 = 9$ $\qquad \cdots \qquad \cdots$ (1)

$$x + y = 3. \qquad \cdots \qquad \cdots \text{ (2)}$$

Writing (1) in the form

$$(x+y)^3 - 3xy(x+y) = 9$$

and substituting the value of $x + y$ from (2), we get

$$27 - 9xy = 9, \qquad \text{or,} \quad xy = 2, \qquad \cdots \text{ (3)}$$

Now, from (2) and (3), we obtain, as in Ex. 4 above

$$x = 2, y = 1 ; \quad \text{or,} \quad x = 1, y = 2.$$

Examples VII(A)

Solve the following equations :

1. (i) $x^2 + y^2 = 1$; $3x + 4y = 5$.

 (ii) $x + y = 2$; $xy = -2$.

2. (i) $x + y = 3$; $2x^2 - 5xy + 2y^2 = 0$.

 (ii) $x + 3y = 7$; $2x^2 + 3xy + 4y^2 = 24$.

3. $x + y = \dfrac{5}{6}$; $\dfrac{1}{x} - \dfrac{1}{y} = 1$.   [*C. U. 1937*]

4. $x + y = 9$; $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}$.

5. (i) $x + y = a + b$; $ax^{-1} + by^{-1} = 2$.

 (ii) $xy + x + y = 27$, $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{2}$.   [*C. U. 1939*]

6. $(a - b) x + (a + b) x = a + b$; $\dfrac{a}{x} + \dfrac{b}{y} = 2a$.

7. $x + y = a + b$; $(x + a)(y + b) = c^2$.

8. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{5}{2}$

 $x + y = 3$.

9. $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = \dfrac{5}{2}$.

 $x + y = 5$.

10. $(x + 2)(y + 3) = 24$

 $xy = 6$.

11. $x + xy = 3$

 $y + xy = 4$.

12. $\dfrac{x}{2} + \dfrac{y}{5} = 5$

 $\dfrac{2}{x} + \dfrac{5}{y} = \dfrac{5}{6}$.

13. $x + \dfrac{1}{y} = a$

 $y + \dfrac{1}{x} = b$.

14. $x^3 - y^3 = 218$

 $x - y = 2$.

15. $x^3 + y^3 = 9$

 $x^2 - xy + y^2 = 3$.

16. $\dfrac{1}{x^2} + \dfrac{1}{y^2} = \dfrac{5}{4}$

      $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{3}{2}.$

17. $\dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{9}{2}$

      $\dfrac{1}{x+y} = \dfrac{1}{3}.$

18. $x^2 + xy + 2y^2 + 3x + 4y = 4$; $x + 2y = 1.$

19. $\dfrac{a}{x} + \dfrac{b}{y} = \dfrac{a^2}{x^2} + \dfrac{b^2}{y^2} = 2.$

20. $x^{\frac{1}{3}} + y^{\frac{1}{3}} = 3$

      $x + y = 9.$

21. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 4$

      $x^{\frac{3}{2}} + y^{\frac{3}{2}} = 28.$

22. $\dfrac{1}{x^2} + \dfrac{1}{xy} + \dfrac{1}{y^2} = \dfrac{7}{9}$

      $\dfrac{1}{x} - \dfrac{1}{y} = \dfrac{5}{3}.$

23. $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{13}{36}$

      $\dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}} = \dfrac{5}{6}.$

24. $x^2 + y^2 + xy = 84$

      $x + y - \sqrt{(xy)} = 6.$

25. $x - \sqrt{x} = 7 - y$

      $y - \sqrt{y} = 9 - x.$

26. $\dfrac{x+y}{1-xy} = 3$; $\dfrac{x-y}{1+xy} = \dfrac{1}{3}.$

27. $(x-y)\dfrac{y}{x} = \dfrac{1}{2}$; $(x-y)\dfrac{x}{y} = 2.$

28. (i) $x^2 + y^2 = 65$; $xy = 28.$

    (ii) $\dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{5}{2}$; $x^2 + y^2 = 90.$

29. $\dfrac{1}{x^2} + \dfrac{1}{xy} = \dfrac{1}{a^2}$

      $\dfrac{1}{y^2} + \dfrac{1}{yx} = \dfrac{1}{b^2}.$

30. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{b}{a} + \dfrac{a}{b}$

      $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = \dfrac{b^2}{a^2} + \dfrac{a^2}{b^2}.$

31. $x^2 + xy + x = 14$

      $y^2 + yx + y = 28.$

32. $x^2 - xy + y^2 = 7$

      $x^4 + x^2 y^2 + y^4 = 133.$

33. $2x^2 + 3xy + y^2 = 70$

      $6x^2 + xy - y^2 = 50.$

34. $3x^2 + 2xy - y^2 = 0$

      $x^2 + y^2 + 2x = 12.$

35. $9^x = 3^y$.   **36.** $8^x . 4^y = 128$
    $8^{xy} = 4^{y+1}$. $9^{x+y} = 27^{xy}$.

37. The area of a rectangular room is 735 sq. ft. and its perimeter is 112 ft. ; find the lengths of its sides.

38. A rectangular courtyard contains 300 square feet and its diagonal is 25 ft. long. Find the length and breadth of the courtyard.

ANSWERS

1. (i) $x = \frac{3}{5}$, $y = \frac{4}{5}$.

(ii) $x = 1 + \sqrt{3}, \ 1 - \sqrt{3}$.
    $y = 1 - \sqrt{3}, \ 1 + \sqrt{3}$.

2. (i) $x = 2, 1$.
    $y = 1, 2$.

(ii) $x = 1, \ -\frac{20}{13}$.
    $y = 2, \ \frac{7}{13}$.

3. $x = \frac{1}{3}, \frac{5}{2}$.
    $y = \frac{3}{2}, \ -\frac{5}{3}$.

4. $x = 3, 6$.
    $y = 6, 3$.

5. (i) $x = a, \frac{1}{2}(a+b)$.
     $y = b, \frac{1}{2}(a+b)$.
    (ii) $x = 3, 6$,
     $y = 6, 3$.

6. $x = \dfrac{a}{a-b}, \dfrac{a+b}{2a}$.
    $y = \dfrac{b}{a+b}, \dfrac{a+b}{2a}$.

7. $x = b + \sqrt{(a+b)^2 - c^2}, \ - \sqrt{(a+b)^2 - c^2}$.
    $y = a - \sqrt{(a+b)^2 - c^2}, \ + \sqrt{(a+b)^2 - c^2}$.

8. $x = 1, 2$.
    $y = 2, 1$.

9. $x = 1, 4$.
    $y = 4, 1$.

10. $x = 2$.
    $y = 3$.

11. $x = 1, -3$.
    $y = 2, -2$.

12. $x = 6, 4$.
    $y = 10, 15$.

13. $x = \dfrac{1}{2b}\{ab + \sqrt{(a^2b^2 - 4ab)}\}, \ \dfrac{1}{2a}\{ab - \sqrt{(a^2b^2 - 4ab)}\}$.

    $y = \dfrac{1}{2a}\{ab + \sqrt{(a^2b^2 - 4ab)}\}, \ \dfrac{1}{2a}\{ab - \sqrt{(a^2b^2 - 4ab)}\}$.

14. $x = 7, -5$.
    $y = 5, -7$.

15. $x = 1, 2$.
    $y = 2, 1$.

16. $x = 1, 2$.
    $y = 2, 1$.

17. $x = 1, 2$.
    $y = 2, 1$.

18. $x = 1, -\frac{3}{2}$.
    $y = 0, \frac{5}{4}$.

19. $x = a$.
    $y = b$.

20. $x = 1, 8.$
$y = 8, 1.$

21. $x = 1, 9.$
$y = 9, 1.$

22. $x = \frac{3}{2}, 1.$
$y = -1, -\frac{3}{2}.$

23. $x = 4, 9.$
$y = 9, 4.$

24. $x = 2, 8.$
$y = 8, 2.$

25. $x = 9.$
$y = 1.$

26. $x = 1, -1.$
$y = \frac{1}{2}, -2.$

27. $x = 2, -\frac{2}{3}.$
$y = 1, \frac{1}{3}.$

28. (i) $x = 4, -4, 7, -7.$
$y = 7, -7, 4, -4.$

(ii) $x = 9, 9, -9, -9.$
$y = 3, -3, 3, -3.$

29. $x = \frac{a}{b} \sqrt{(a^2 + b^2)}, -\frac{a}{b} \sqrt{(a^2 + b^2)}.$
$y = \frac{b}{a} \sqrt{(a^2 + b)}, -\frac{b}{a} \sqrt{(a^2 + b^2)}.$

30. $x = b, \dfrac{a^2}{b}.$
$y = a, \dfrac{b^2}{a}.$

31. $x = 2, -\frac{7}{3}.$
$y = 4, -\frac{14}{5}.$

32. $x = 2, 3, -2, -3.$
$y = 3, 2, -3, -2.$

33. $x = 3, -3.$
$y = 4, -4.$

34. $x = 1, 2, -3, -\frac{6}{5}.$
$y = 3, -2, 3, -\frac{18}{5}.$

35. $x = 1, -\frac{1}{3}.$
$y = 2, -\frac{2}{3}.$

36. $x = 1, \frac{14}{9}.$
$y = 2, \frac{7}{6}.$

37. Length $= 35$ ft., breadth $= 21$ ft.

38. 20 ft., 15 ft.

SEC. B. THREE UNKNOWN QUANTITIES

39. No general rule can be laid down for the solution of quadratic equations involving three unknown quantities. The following examples will serve to indicate the various artifices to be employed in different cases.

Ex. 1. *Solve* $x - 2y + z = 0$ $\qquad \cdots \qquad \cdots$ (1)
$9x - 8y + 3z = 0$ $\qquad \cdots \qquad \cdots$ (2)
$xy + xz + yz = 23.$ $\qquad \cdots \qquad \cdots$ (3)

From (1) and (2), by Cross Multiplication,

$$\frac{x}{-6+8} = \frac{y}{9-3} = \frac{z}{-3+18}.$$

or, $\quad \dfrac{x}{1} = \dfrac{y}{3} = \dfrac{z}{5} = k$ suppose.

$\therefore \quad x = k, \quad y = 3k, \quad z = 5k.$

Substituting these values of x, y, z in (3), we get

$3k^2 + 5k^2 + 15k^2 = 23,$ $\qquad$ or, $\quad k^2 = 1,$ *i.e.*, $k = \pm 1.$

$\therefore \quad x = 1, y = 3, z = 5$; or, $x = -1, y = -3, z = -5.$

Ex. 2. *Solve* $x + y + 2z = 9$ ...   ... (1)

$$x^2 + y^2 + z^2 = 14$$ ...   ... (2)

$$yz + zx + xy = 11$$ ...   ... (3)

for real values of x, y, z.

Since, $(x + y + z)^2 = x^2 + y^2 + z^2 + 2(yz + zx + xy) = 14 + 22 = 36.$

∴ $x + y + z = \pm 6$ ...   . (4)

Subtracting (4) from (1), we get $z = 3$, or, 15.

From (2), it is clear that the value of $z = 15$ inadmissible.

Since, $z = 3$, ∴ from (1), $x + y = 3$ }   ... (5)
and from (2), $x^2 + y^2 = 5$ }

∴ Solving equations (5) as in Ex. 1, Art. 37, we shall get

$$x = 1, \text{ or, } 2 \text{ ; and } y = 2, \text{ or, } 1,$$

∴   The roots are $x = 1, y = 2, z = 3$;

and $x = 2, y = 1, z = 3.$

Ex. 3. *Solve* $x(y + z) = 5,\ y(z + x) = 8,\ z(x + y) = 9.$

[*C. U. 1938*]

From the above three equations, we have

$$xy + xz = 5 \quad \cdots \qquad \cdots \qquad \cdots \quad (1)$$

$$yz + yx = 8 \quad \cdots \qquad \cdots \qquad \cdots \quad (2)$$

$$zx + zy = 9. \quad \cdots \qquad \cdots \qquad \cdots \quad (3)$$

Adding (1), (2) and (3) and dividing by 2, we get

$$yz + zx + xy = 11. \qquad \cdots \qquad \cdots \quad (4)$$

Subtracting (1), (2), (3) successively from (4), we get

$$yz = 6,\ zx = 3,\ xy = 2. \qquad \cdots \qquad \cdots \quad (5)$$

∴ $yz \cdot zx \cdot xy = 6 \cdot 3 \cdot 2$: or, $x^2 y^2 y^2 = 36.$

∴ $xyz = \pm 6. \qquad \cdots \qquad \cdots \qquad \cdots \quad (6)$

Dividing (6) by the three relations (5) successively, we get

$$x = \pm 1 \text{ ; } y = \pm 2 \text{ ; } z = \pm 3.$$

Hence, the roots of the equations are

$$x = 1\ y = 2,\ z = 3 \text{ and } x = -1,\ y = -2,\ z = -3.$$

Ex. 4. *Solve* $x^2 + xy + xz = 9$ ...   ... (1)

$$y^2 + yz + yx = 27$$ ...   ... (2)

$$z^2 + zx + zy = 45$$ ...   ... (3)

Adding up the three equations, we have

$$x^2 + y^2 + z^2 + 2(yz + zx + xy) = 81.$$

∴ $(x + y + z)^2 = 81.$ ∴ $x + y + z = \pm 9.$ ... (4)

Since, (1) is $x(x+y+z)=9$.

$\therefore$ Dividing (1) by (4), we have, $x=\pm 1$.

Similarly dividing (2) and (3) by (4), we have, $y=\pm 3$, $z=\pm 5$.

$\therefore$ The roots are $x=1$, $y=3$, $z=5$ and $x=-1$, $y=-3$, $z=-5$.

Examples VII(B)

Solve the following equations :

1. $3x+y-5z \quad = 0$
$7x-3y-9z \quad = 0$
$x^2+2y^2+3z^2 \quad = 23.$

2. $x+y+z \quad = 13$
$x^2+y^2+z^2 = 91$
$\qquad\qquad y^2 = zx.$

3. $x+y+z \quad = 10$
$x^2+y^2+z^2 = 38$
$\qquad\qquad yz = 15.$

4. $xz+y \quad = 7z$
$yz+x \quad = 8z$
$x+y+z = 12.$

5. $2x+3y = 5xy$
$3y+4z = 7yz$
$4z+5x = 9zx.$

6. $yz = 6$
$zx = 3$
$xy = 2.$

7. $x(y+z) = 44$
$y(z+x) = 50$
$z(x+y) = 54.$

8. $xy+x+y = 23$
$yz+y+z = 27$
$zx+z+x = 41.$
             [*C. U. 1945*]

9. $(x+y)(x+z) = 48$
$(y+z)(y+x) = 60$
$(z+x)(z+y) = 80.$

10. $xy = x+y$
$yz = 2(y+z)$
$zx = 3(z+x).$

11. $xyz = \frac{6}{5}(y+z) = \frac{3}{2}(z+x) = 2(x+y).$

12. $(y+z):(z+x):(x+y) = a:b:c$
$(y+z)^2+(z+x)^2+(x+y)^2 = 1.$

13. $xy+2(x+y) = 20$
$yz+2(y+z) = 38$
$zx+2(z+x) = 24.$

14. $y+z = x^{-1}$
$z+x = y^{-1}$
$x+y = z^{-1}.$
             [*C. U. 1944*]

15. $\dfrac{y+z}{3} = \dfrac{z+x}{5} = \dfrac{x+y}{6}$; $x^2 + y^2 + z^2 = 84$.

16. $(y+z)(x+y+z) = \frac{7}{2}$ **17.** $x(1+y) = 3$

$(z+x)(x+y+z) = 2$ $y(1+z) = 8$

$(x+y)(x+y+z) = \frac{5}{2}$. $z(1+x) = 6$. [$C.\ U.\ 1943$]

18. $x^2 - yz = 5$, $y^2 - zx = 3$, $z^2 - xy = -1$.   [$C.\ U.\ 1946$]

19. $2x + 25y + 7 = 10xy$ **20.** $x + y^{-1} = \frac{8}{3}$

$5y + 3z + 31 = 15yz$ $y + z^{-1} = \frac{7}{3}$

$15z + 2x + 19 = 6zx$. $z + x^{-1} = 4$ [$C.\ U.\ 1942$]

21. $\dfrac{yz}{ny+mz} = \dfrac{zx}{lz+nx} = \dfrac{xy}{mx+ly}$; $x+y+z = 1$.

22. The area of the floor of a room is 320 sq. ft. and those of the two adjacent walls are 200 sq. ft. and 160 sq. ft. respectively. Find the length, breadth and height of the room.

23. The sum of the three sides of a right-angled triangle is 30 and the product of these sides is 780 ; find the lengths of the sides.

ANSWERS

1. $x = 3,\ -3$.
$y = 1,\ -1$.
$z = 2,\ -2$.

2. $x = 9,\ 1$.
$y = 3,\ 3$.
$z = 1,\ 9$.

3. $x = 2,\ 2,\ 8,\ 8$.
$y = 3,\ 5,\ 1 - \sqrt{-14},\ 1 + \sqrt{-14}$.
$z = 5,\ 3,\ 1 + \sqrt{-14},\ 1 - \sqrt{-14}$.

4. $x = 4,\ \frac{60}{7}$.
$y = 4,\ \frac{66}{7}$.
$z = 2,\ -6$.

5. $x = 0,\ 1$.
$y = 0,\ 1$.
$z = 0,\ 1$.

6. $x = 1,\ -1$.
$y = 2,\ -2$.
$z = 3,\ -3$.

7. $x = 4,\ -6$.
$y = 5,\ -5$.
$z = 6,\ -6$.

8. $x = 5,\ -7$.
$y = 3,\ -5$.
$z = 6,\ -8$.

9. $x = 2,\ -2$.
$y = 4,\ -4$.
$z = 6,\ -6$.

10. $x = 0,\ \frac{12}{5}$.
$y = 0,\ \frac{12}{7}$.
$z = 0,\ -12$.

11. $x = 0,\ 1,\ -1$.
$y = 0,\ 2,\ -2$.
$z = 0,\ 3,\ -3$.

12. $\dfrac{x}{b+c-a} = \dfrac{y}{c+a-b} = \dfrac{z}{a+b-c} = \pm \dfrac{1}{2\sqrt{a^2+b^2+c^2}}$.

13. $x = 2,\ -6$.
$y = 4,\ -8$.
$z = 5,\ -9$.

14. $x = y = z = \pm \dfrac{1}{\sqrt{2}}$.

15. $\dfrac{x}{4} = \dfrac{y}{2} = \dfrac{z}{1} = \pm 2$.

16. $x = \frac{1}{4}, -\frac{1}{4}.$
$y = 1, -1.$
$z = \frac{3}{4}. -\frac{3}{4}.$

17. $x = 1, -\frac{7}{3}.$
$y = 2, -\frac{16}{7}.$
$z = 3, -\frac{9}{2}.$

18. $x = 2, -2.$
$y = 1, -1.$
$z = -1, 1.$

19. $x = 4, 1.$
$y = 1, -\frac{3}{5}.$
$z = 3, -\frac{7}{3}.$

20. $x = 1, \frac{3}{10}.$
$y = 2, \frac{5}{6}.$
$z = 3, \frac{2}{3}.$

21. $x = \dfrac{l}{l+m+n}.$

$y = \dfrac{m}{l+m+n}.$

$z = \dfrac{n}{l+m+n}.$

22. length $= 20$ ft., breadth $= 16$ ft., height $= 10$ ft.   **23.** $5, 12, 13.$

Examples VII(C)

[*Additional Examples on Quadratic Equations*]

Solve the following Equations :

1. $\dfrac{x^2 - a^2 - b^2}{c^2} + \dfrac{c^2}{x^2 - a^2 - b^2} = 2.$

2. $(x - a + 2b)^3 - (x - 2a + b)^3 = (a + b)^3.$

3. $\sqrt[3]{15 - x} + \sqrt[3]{x + 57} = 6.$ **4.** $\dfrac{x-a}{b} + \dfrac{x-b}{a} = \dfrac{b}{x-a} + \dfrac{a}{x-b}.$

5. $\sqrt{3x-1} + \sqrt{5x-2} = \sqrt{7x-3} + \sqrt{9x-4}.$

6. $2^{2x+3} - 57 = 65\,(2^x - 1).$ **7.** $x^x - x^{-x} = 3\,(1 + x^{-x}).$

8. $4x^2 - 2xy + 5y^2 + 4x - 11y - 14 = 0,\ 3y - 2x + 1 = 0.$

9. $\dfrac{1}{x^3} - \dfrac{1}{y^3} = 19$

$\dfrac{1}{x} - \dfrac{1}{y} = 1.$

10. $\dfrac{12}{x+y} + \dfrac{3}{x} = 4$

$\dfrac{12}{x-y} + \dfrac{3}{x} = 7.$

11. $x^2 + y^2 + x + y = 26$
$\qquad 4(x + y) = 3xy.$

12. $xy + x + y = 27$
$\qquad 2(x^{-1} + y^{-1}) = 1.$

13. $x + y + xy = 5$
$\qquad xy\,(x + y) = 6.$

14. $(x - 3)(y - 2) = 6$
$\qquad (2x + 1)(y - 1) = -12.$

15. $\dfrac{x}{y} + \dfrac{y}{x} = \dfrac{5}{2},\ \dfrac{x^2}{y} + \dfrac{y^2}{x} = \dfrac{9}{2}.$

16. $xy - x^2 = 1,\ y^2 - xy = 2.$

17. $\sqrt{x+y} + \sqrt{x-y} = 4,\ x^2 - y^2 = 9.$

18. $xy + 6 = 2x - x^2,\ xy - 9 = 2y - y^2.$

19. $\dfrac{\sqrt[3]{x+y} - \sqrt[3]{x-y}}{\sqrt[3]{x+y} + \sqrt[3]{x-y}} = 3,\ x^2 + y^2 = 65.$

20. $x^y = y^2,\ y^{2y} = x^4.$ [C. U. 1941, '45]

21. $y^x = 4,\ y^2 = 2^x.$ [C. U. 1943]

22. $18y^x - y^{2x} = 81,\ y^2 = 3^x.$ **23.** $x = y = z = xyz.$

24. $x^2 + y^2 + z^2 = 84,\ x + y + z = 14,\ xz = y^2.$

25. $2y - yz = 4,\ 2z - zx = 9,\ 2x - xy = 16.$

26. $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c} = \dfrac{xyz}{x+y+z}.$

27. $\dfrac{x}{y} + \dfrac{y}{z} + \dfrac{z}{x} = \dfrac{y}{x} + \dfrac{z}{y} + \dfrac{x}{z} = x + y + z = 3.$

28. $x(x^2 + y^2 + z^2) = 14$
$y(x^2 + y^2 + z^2) = 28$
$z(x^2 + y^2 + z^2) = 42.$

29. $yz + zx - xy = -1$
$zx + xy - yz = 7$
$xy + yz - zx = -11.$

30. $x + y + z = 4$
$x + 3y + 2z = 11$
$yz + 4zx + 2xy = 0.$

31. $yz(y - z) = -30$
$zx(z - x) = 30$
$xy(x - y) = 6.$

32. $x^2 - (y - z)^2 = a^2$
$y^2 - (z - x)^2 = b^2$
$z^2 - (x - y)^2 = c^2.$

33. $4x - 2y - z = 0$
$6x - y - 4z = 0$
$xy - yz + zx = 46.$

34. $x^2 - y^2 - z^2 + 2yz = 3,\ y^2 - z^2 - x^2 + 2zx = 5$ and
$z^2 - x^2 - y^2 + 2xy = 15.$

35. If the equations $ax + by = 1,\ cx^2 + dy^2 = 1$ have two identical solutions, prove that $\dfrac{a^2}{c} + \dfrac{b^2}{d} = 1$ and $x = \dfrac{a}{c},\ y = \dfrac{b}{d}.$

ANSWERS

1. $\pm\sqrt{a^2+b^2+c^2}$. 2. $a-2b,\ 2a-b$. 3. $7,\ -49$.

4. $0,\ a+b,\ \dfrac{a^2+b^2}{a+b}$. 5. $\frac{1}{2}$. 6. ±3. 7. $2,\ -1$.

8. $x=2,\ -1$. 9. $x=\frac{1}{3},\ -\frac{1}{2}$. 10. $x=3,\ \frac{15}{28}$.
$\ \ \ y=1,\ -1$. $\ \ \ y=\frac{1}{2},\ -\frac{1}{3}$. $\ \ \ y=1,\ -2\frac{35}{28}$.

11. $x=4,\ 2,\ \frac{1}{6}(-13\pm\sqrt{377})$. 12. $x=3,\ 6$.
$\ \ \ y=2,\ 4,\ \frac{1}{6}(-13\mp\sqrt{377})$. $\ \ \ \ \ y=6,\ 3$.

13. $x=1,\ 2,\ 1+\sqrt{-2},\ 1-\sqrt{-2}$. 14. $x=\frac{3}{2},\ 11$. 15. $x=2,\ 1$.
$\ \ \ y=2,\ 1,\ 1-\sqrt{-2},\ 1+\sqrt{-2}$. $\ \ \ y=-2,\ \frac{11}{7}$. $\ \ \ y=1,\ 2$.

16. $x=1,\ -1$. 17. $x=5,\ 5$. 18. $x=-6,\ 2$. 19. $x=\dfrac{7}{\sqrt{2}},\ -\dfrac{7}{\sqrt{2}}$.
$\ \ \ y=2,\ -2$. $\ \ \ y=4,\ -4$. $\ \ \ y=9,\ -3$. $\ \ \ \ \ \ y=\dfrac{9}{\sqrt{2}},\ -\dfrac{9}{\sqrt{2}}$.

20. $x=\pm2,\ \pm\frac{1}{2}$. 21. $x=2,\ -2$. 22. $x=2,\ -2$.
$\ \ \ y=2,\ -2$. $\ \ \ y=\pm2,\ \pm\frac{1}{2}$. $\ \ \ y=\pm3,\ \pm\frac{1}{3}$.

23. $x=0,\ 1,\ -1$. 24. $x=8,\ 2$. 25. $x=-\frac{40}{7}$.
$\ \ \ y=0,\ 1,\ -1$. $\ \ \ y=4,\ 4$. $\ \ \ y=\frac{24}{5}$.
$\ \ \ z=0,\ 1,\ -1$. $\ \ \ z=2,\ 8$. $\ \ \ z=\frac{7}{6}$.

26. $x=\dfrac{a+b+c}{bc}$. 27. $x=1$. 28. $x=1,\ \omega,\ \omega^2$.
$\ \ \ y=\dfrac{a+b+c}{ca}$. $\ \ \ y=1$. $\ \ \ y=2\ 2\omega,\ 2\omega^2$.
$\ \ \ z=\dfrac{a+b+c}{ab}$. $\ \ \ z=1$. $\ \ \ z=3,\ 3\omega,\ 3\omega^2$.

29. $x=1,\ -1$. 30. $x=1,\ -\frac{3}{2}$. 31. $x=3,\ 3\omega,\ 3\omega^2$.
$\ \ \ y=-2,\ 2$. $\ \ \ y=4,\ 2\frac{1}{5}$. $\ \ \ y=2,\ 2\omega,\ 2\omega^2$.
$\ \ \ z=3,\ -3$. $\ \ \ z=-1,\ \frac{7}{4}$. $\ \ \ z=5,\ 5\omega,\ 5\omega^2$.

32. $x=\dfrac{a(b^2+c^2)}{2bc},\ -\dfrac{a(b^2+c^2)}{2bc}$. 33. $x=7,\ -7$. 34. $x=2,\ -2$.
$\ \ \ y=\dfrac{b(c^2+a^2)}{2ca},\ -\dfrac{b(c^2+a^2)}{2ca}$. $\ \ \ y=10,\ -10$. $\ \ \ y=3,\ -3$.
$\ \ \ z=\dfrac{c(a^2+b^2)}{2ab},\ -\dfrac{c(a^2+b^2)}{2ab}$. $\ \ \ z=8,\ -8$. $\ \ \ z=4,\ -4$.

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THEORY OF QUADRATIC EQUATIONS AND EXPRESSIONS

40. Number of Roots of a Quadratic.

It has already been found in Art. 35 that the most general form of a quadratic equation has two roots. It will now be shown that

A quadratic equation has two and only two roots.

Let a be a root[*] of the equation $ax^2 + bx + c = 0$.

Then $aa^2 + ba + c = 0$. $\therefore$ $c = -(aa^2 + ba)$.

$\therefore$ $ax^2 + bx + c = ax^2 + bx - aa^2 - ba$
$$= a(x^2 - a^2) + b(x - a)$$
$$= (x - a)\{a(x + a) + b\}. \qquad \cdots \quad (1)$$

Thus, we see that corresponding to a root a of the equation $ax^2 + bx + c = 0$, there is a linear factor $x - a$ of
$$ax^2 + bx + c.$$

Since, from (1) it is clear that $ax^2 + bx + c$ has got two and only two linear factors of the form $x - a$, (real or imaginary, distinct or coincident), it follows that the quadratic equation $ax^2 + bx + c = 0$ has got two and only two roots (real or imaginary, distinct or coincident).

Note. We give below a proof which is instructive to show that *a quadratic equation have more than two different roots.*

If possible, let a, β, γ be three *different* values of the unknown quantity x, which satisfy the standard quadratic equation
$$ax^2 + bx + c = 0 \qquad \cdots \qquad \cdots \quad (1)$$

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[*]Here it is tacitly assumed that every equation has a root.

Since each of these satisfies the equation, we must have

$$a\alpha^2 + b\alpha + c = 0 \qquad \cdots \qquad \cdots \quad (2)$$
$$a\beta^2 + b\beta + c = 0 \qquad \cdots \qquad \cdots \quad (3)$$
$$a\gamma^2 + b\gamma + c = 0. \qquad \cdots \qquad \cdots \quad (4)$$

Subtracting (3) from (2), we get

$$a(\alpha^2 - \beta^2) + b(\alpha - \beta) = 0,$$

or, $(\alpha - \beta)\{a(\alpha + \beta) + b\} = 0.$

Now, since α is different from β, $(\alpha - \beta)$ cannot be equal to zero,

$$\therefore \quad a(\alpha + \beta) + b = 0. \quad \cdots \qquad \cdots \quad (5)$$

Similarly, subtracting (4) from (3), we get

$$a(\beta + \gamma) + b = 0. \qquad \cdots \qquad \cdots \quad (6)$$

Subtracting (6) from (5), $a(\alpha - \gamma) = 0$ $\cdots \quad (7)$

which is impossible, since by hypothesis, $a \neq 0$ and $\alpha \neq \gamma$.

Hence, a quadratic equation cannot have more than two different roots.

Note. Now, if we *assume* that the quadratic equation $ax^2 + bx + c = 0$ is satisfied by three different values of x, say, α, β, γ then proceeding as above, we shall obtain from relation (7) that $a = 0$, since $(\alpha - \gamma) \neq 0$ and hence from (6), we have $b = 0$ and from (4), $c = 0$. Thus, the equation becomes $0.x^2 + 0.x + 0 = 0$ and this being true for all values of x, is an *identity*. Thus, *if a quadratic equation is satisfied by more than two different values of the unknown quantity, it is an identity and not an equation.*

41. Nature of the roots of a Quadratic.

It will now be shown how without solving a quadratic, the nature of its roots can be determined. We take the typical equation $ax^2 + bx + c = 0$, the roots of which are

$$-\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a} \text{ and } -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}$$

and we suppose that a, b, c are *real* quantities. The expression $b^2 - 4ac$ enters into both the solutions. Hence, the

nature of the roots of the quadratic depending upon the value of $b^2 - 4ac$ will be considered.

(i) If $b^2 - 4ac$ is *positive, i.e.,* $b^2 > 4ac$, then $\sqrt{b^2 - 4ac}$ is real and therefore both the roots are *real* and *unequal.* In particular, if $b^2 - 4ac$ is *positive* and a *perfect square*, both the roots are *rational*, provided a, b, c are rational ; and if $b^2 - 4ac$ is *positive* but *not a perfect square*, they are *irrational.*

(ii) If $b^2 - 4ac$ is *zero, i.e.,* $b^2 = 4ac$, both the roots are *real* and *equal.*

(iii) If $b^2 - 4ac$ is *negative, i.e.,* $b^2 < 4ac$, both the roots are *imaginary* and *unequal.*

Note 1. From above, it is clear that *both* the *roots are real* if $b^2 - 4ac$ is positive or zero, *i.e.,* if $b^2 - 4ac$ *is not negative.*

Note 2. The quantity $b^2 - 4ac$ is called the *Discriminant* of the quadratic equation $ax^2 + bx + c = 0$ (or of the corresponding quadratic expression), since it is so very useful in discriminating the nature of its roots.

Note 3. From (i) it is clear that a quadratic with rational coefficients cannot have one rational and another irrational root and from (iii) it follows that a quadratic with real coefficients cannot have one real and another imaginary root.

42. Conjugate Roots.

(i) *In a quadratic equation with rational coefficients, irrational roots occur in pairs, i.e.,* if $p + \sqrt{q}$ be one root, the other root will be the conjugate irrational quantity $p - \sqrt{q}$ and conversely.

Let the irrational quantity $p + \sqrt{q}$ be a root of the quadratic $ax^2 + bx + c = 0$. Then,

$$a(p + \sqrt{q})^2 + b(p + \sqrt{q}) + c = 0,$$

or, $(ap^2 + aq + bp + c) + \sqrt{q}(2ap + b) = 0.$

Since rational and irrational parts cannot cancel each other,   [See *Art. 18 (ii)*],

$\therefore \quad ap^2 + aq + bp + c = 0$ and $2ap + b = 0.$ $\cdots$ (1)

Now, $a(p - \sqrt{q})^2 + b(p - \sqrt{q}) + c$

$$= (ap^2 + aq + bp + c) - \sqrt{q}\,(2ap + b)$$

$$= 0 - \sqrt{q}.0 \text{ by } (1), = 0.$$

$\therefore \quad p - \sqrt{q}$ is also a root of $ax^2 + bx + c = 0.$

(ii) *In a quadratic equation with real coefficients, imaginary (complex) roots occur in pairs, i.e.,* if $p + iq$ be one root, the other root will be the conjugate imaginary quantity $p - iq$ and conversely.

Let the imaginary quantity $p + iq$ be a root of the quadratic

$$ax^2 + bx + c = 0.$$

Then, $a(p + iq)^2 + b(p + iq) + c = 0,$

or, $(ap^2 - aq^2 + bp + c) + i(2apq + bq) = 0.$

$\therefore$ Equating real and imaginary parts, [See *Art. 25(ii)*]

$$ap^2 - aq^2 + bp + c = 0 \text{ and } 2apq + bq = 0. \quad \cdots \quad (1)$$

Now, $a(p - iq)^2 + b(p - iq) + c$

$$= (ap^2 - aq^2 + bp + c) - i(2apq + bq)$$

$$= 0 - i.0, \text{ by } (1), = 0.$$

$\therefore \quad p - iq$ is also a root of $ax^2 + bx + c = 0.$

43. Relation between Roots and Coefficients.

Let a, β be the roots of the quadratic $ax^2 + bx + c = 0$; then from Art. 35, it follows that

$$a = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

$\therefore$ (i) $\alpha + \beta = -\dfrac{2b}{2a} = -\dfrac{b}{a}$.

(ii) $\alpha\beta = \dfrac{(-b)^2 - (\sqrt{b^2 - 4ac})^2}{4a^2} = \dfrac{b^2 - (b^2 - 4ac)}{4a^2}$

$\qquad = \dfrac{4ac}{4a^2} = \dfrac{c}{a}$.

Thus,

(i) *Sum of the roots* $= -\dfrac{b}{a} = -\dfrac{\text{coefficient of } x}{\text{coefficient of } x^2}$.

(ii) *Product of the roots* $= \dfrac{\text{absolute term}}{\text{coefficient of } x^2}$.

Cor. If the equation be written in the form $x^2 + \dfrac{b}{a}x + \dfrac{c}{a} = 0$, so that the coefficient of x^2 is unity, the above relation may be stated thus :—

In a quadratic equation with coefficient of x^2 as unity,

  (i) *the sum of the roots is equal to the coefficient of x with the sign changed,*

  (ii) *the product of roots is equal to the absolute term.*

Note. It evidently follows from above that the quadratic equation, of which the sum and the product of the roots are p and q respectively, if $x^2 - px + q = 0$.

44. Formation of a quadratic whose roots are given.

Let the roots α, β of a quadratic be given. Suppose

$\qquad x^2 + px + q = 0$ be the required quadratic equation.

Then $\alpha + \beta = -p$ and $\alpha\beta = q$.

$\therefore$ The required quadratic equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0 \qquad\qquad \cdots \ (1)$$

or, $(x - \alpha)(x - \beta) = 0$.

The equation (1) can be written in words thus :

$$x^2 - (\textit{sum of the roots})\, x + \textit{product of the roots} = 0.$$

45. Illustrative Examples.

Ex. 1. *For what values of a will the equation* $x^2 - (3a-1)\, x + 2a^2 + 2a - 11 = 0$ *have equal roots ?*

Here the discriminant must be zero.

∴ $(3a-1)^2 - 4(2a^2 + 2a - 11) = 0,$

or, $a^2 - 14a + 45 = 0,$

or, $(a-5)(a-9) = 0$;   ∴ $a = 5,$ or, 9.

Ex. 2. *If a, b, c are rational and* $a + b + c = 0,$ *show that the roots of the equation*

$$(b+c-a)\, x^2 + (c+a-b)\, x + (a+b-c) = 0 \textit{ are rational.}$$

Here, the discriminant

$$= (c+a-b)^2 - 4(b+c-a)(a+b-c)$$
$$= (-a-b+a-b)^2 - 4(b-a-b-a)(a+b+a+b)$$

$$[\text{ since } c = -(a+b)]$$

$$= 4b^2 + 16a(a+b)$$
$$= 4\{b^2 + 4a^2 + 4ab\} = 4(2a+b)^2 = \text{a perfect square,}$$

and since a, b are rational, therefore the roots are rational.

N. B. The equation reduces to $ax^2 + bx + c = 0$;   ∴   Discriminant $= b^2 - 4a(-a-b) = (2a+b)^2 = $ a perfect square.

Ex. 3. *Find the equation with rational coefficient, one of whose roots is* $3 + \sqrt{5}.$

Since one root is $3 + \sqrt{5},$ the other root must be $3 - \sqrt{5}$ [*Art. 42(i)*]

∴   Sum of the roots $= (3 + \sqrt{5}) + (3 - \sqrt{5}) = 6,$

product of the roots $= (3 + \sqrt{5})(3 - \sqrt{5}) = 9 - 5 = 4$;

∴   The reqd. equation is $x^2 - 6x + 4 = 0.$ [*Art. 44*]

Ex. 4. *Find the quadratic equation with real coefficients, one of whose roots is $5 - \sqrt{-3}$.*

The other root must be $5 + \sqrt{-3}$.   [*Art. 42(ii)*]

$\therefore$ *Sum* of the roots $= 10$ and their *product* $= 25 + 3 = 28$;

$\therefore$ The reqd. equation is $x^2 - 10x + 28 = 0$.

Ex. 5. *Find the value of $x^3 - 7x^2 - 2x + 88$ when $x = 5 + \sqrt{-3}$.*

The equation whose roots are $5 + \sqrt{-3}$ and $5 - \sqrt{-3}$ is
$$x^2 - 10x + 28 = 0. \qquad\qquad [\text{ See } Ex.\ 4\ above\ ]$$

Hence the value of $x^2 - 10x + 28$ is 0, when $x = 5 + \sqrt{(-3)}$.

Given expression $= x\,(x^2 - 10x + 28) + 3\,(x^2 - 10x + 28) + 4$
$$= x \times 0 + 3 \times 0 + 4 = 4.$$

Note. In examples of this type, *i.e.*, in the evaluation of expressions containing x when x is of the form $a \pm \sqrt{b}$ or $a \pm \sqrt{(-b)}$, the method of procedure is "First form the quadratic, one of whose roots is the given value of x"; then with the help of this expression, the evaluation of the given expression becomes easier.

Ex. 6. *If r be the ratio of the roots of the equation $ax^2 + bx + c = 0$, show that $\dfrac{(r+1)^2}{r} = \dfrac{b^2}{ac}.$*   [*C. U. 1934*]

Let α be one of the roots, so that $r\alpha$ is the other root.

Then $\alpha + r\alpha = -\dfrac{b}{a}$,   *i.e.*, $\alpha(1 + r) = -\dfrac{b}{a}$ $\cdots$ $\cdots$ (1)

and $\alpha.r\alpha = \dfrac{c}{a}$,   *i.e.*, $\alpha^2 r = \dfrac{c}{a}$. $\cdots$ $\cdots$ (2)

From (1), $\alpha^2 = \dfrac{b^2}{a^2(1+r)^2}$;   from (2), $\alpha^2 = \dfrac{c}{ar}$;

$\therefore$ $\dfrac{b^2}{a^2(1+r)^2} = \dfrac{c}{ar}$,   *i.e.*, $\dfrac{(r+1)^2}{r} = \dfrac{b^2}{ac}$.

Ex. 7. *Find the condition that the roots of the quadratic equation $ax^2 + bx + c = 0$ should be*

     (i) *equal in magnitude and opposite in sign,*

and   (ii) *reciprocals.*

(i) If the roots be equal in magnitude and opposite in sign, then their sum would be zero.

$\therefore$ Sum of roots $= -\dfrac{b}{a} = 0$, or, $b = 0$.

$\therefore$ $b = 0$ is the required condition.

(ii) Let the roots be a and $\dfrac{1}{a}$.

$\dfrac{c}{a}$ = product of the roots $= a \times \dfrac{1}{a} = 1$.

$\therefore$ $c = a$ is the required condition.

Ex. 8. *Find the condition that the roots of the quadratic equation* $ax^2 + bx + c = 0$ *should be*

(i) *both positive*

(ii) *both negative*

(iii) *one positive* and *the other negative.*

(i) If the roots are both positive, $a + \beta$ and $a\beta$ must be both positive, *i.e.,* $-b/a$ and c/a are both positive.

Hence, the required condition is that the signs of a and c should be like and opposite to that of b.

(ii) If the roots are both negative. $a + \beta$ is negative, and $a\beta$ is positive, *i.e.,* $-b/a$ is negative and c/a is positive.

Hence, the required condition is a, b, c must all be of the same sign.

(iii) If one of the roots is positive and the other negative, $a\beta$ must be negative. $\therefore$ c/a is negative, *i.e.,* c and a must be of opposite signs.

The numerically greater root will be positive or negative according as the sum of the roots is positive or negative, *i.e.,* according as b/a is negative or positive.

Hence *the numerically greater root is positive* if b and c are of the same signs and a of the opposite sign and *is negative* if a and b are of the same signs and c of the opposite sign.

Ex. 9. *Find the condition that in the quadratic* $ax^2 + bx + c = 0$

(i) *one root is zero,*

(ii) *both the roots are zero.*

(i) If one root is zero, c/a = product of the roots = 0 ;

∴ The required condition is $c = 0$.

(ii) If both the roots are zero, sum of the roots as well as the product of the roots are zero ; ∴ $-b/a = 0$ and $c/a = 0$;

∴ The reqd. condition is $b = c = 0$.

Ex. 10. (i) Find the condition that the two equations $ax^2 + bx + c = 0$ **and** $a'x^2 + b'x + c' = 0$ **may have one root common.** *Assuming that this condition is satisfied, find the common root and also the other roots of the equations.*

(ii) *What is the condition that the above two quadratics should have both the roots common ?*

(i) Let α be the common root; since it satisfies both the equations.

∴ $a\alpha^2 + b\alpha + c = 0,$

$a'\alpha^2 + b'\alpha + c' = 0.$

∴ By the rule of Cross-Multiplication,

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{ca' - c'a} = \frac{1}{ab' - a'b}. \qquad \cdots \qquad \cdots \quad (1)$$

∴ $\alpha = \dfrac{bc' - b'c}{ca' - c'a}$ and also $= \dfrac{ca' - c'a}{ab' - a'b}. \qquad \cdots \qquad \cdots \quad (2)$

∴ $(bc' - b'c)(ab' - a'b) = (ca' - c'a)^2. \qquad \cdots \qquad \cdots \quad (3)$

This is the required condition.

From (2), it follows that the *common root* α is

$$\frac{bc' - b'c}{ca' - c'a} \text{ or } \frac{ca' - c'a}{ab' - a'b}$$

which are obviously equal because of the relation (3).

Since, the product of the two roots of the 1st equation = c/a.

∴ The *other root* of the 1st equation, obtained by dividing c/a by either value of the common root, is

$$\frac{c\,(ca'-c'a)}{a\,(bc'-b'c)} \quad \text{or} \quad \frac{c\,(ab'-a'b)}{a\,(ca'-c'a)}.$$

Similarly, the *other root* of the 2nd equation can be obtained.

(ii) Let a, β be the roots of the 1st equation and a', β', those of the 2nd equation ; since both the roots are common to the two equations,

∴ $a+\beta=a'+\beta'$ and $a\beta=a'\beta'$,

∴ $-\dfrac{b}{a}=-\dfrac{b'}{a'}$ and $\dfrac{c}{a}=\dfrac{c'}{a'}$,

i.e., $\dfrac{a}{a'}=\dfrac{b}{b'}=\dfrac{c}{c'}.$

This is the required condition.

Note. It is easily seen that in case (ii) the two equations are equivalent.

Examples VIII(A)

In the following examples letters a, b, c etc. denote real quantities unless otherwise stated.

1.   Is the following an equation or identity ?

(i) $(x^2-a)(b-a)+(x^2-b)(a-b)=(a-b)^2.$

(ii) $(x-m)^2+(x-n)^2$

$$=x\,(x-m)+x\,(x-n)+m\,(m-x)+n\,(n-x).$$

2.   Show that the roots of the equation

$$(b-cx)^2+2\,(c-a)\,x+(a-b)=0$$

are always real.

3.   Show that the roots of $ax^2+bx+c=0$ are rational, if $a+b+c=0$ and if a, b, c are rational.

4. If the roots of the equation

$$(a^2 + b^2)\, x^2 - 2\,(ac + bd)\, x + (c^2 + d^2) = 0$$

are equal, then $a : b :: c : d$.

5. Show that the roots of the equation

$$(a^2 - bc)\, x^2 + 2\,(b^2 - ca)\, x + (c^2 - ab) = 0$$

will be equal either $b = 0$, or $a^3 + b^3 + c^3 - 3abc = 0$.

6. For what values of m will the equation

$$x^2 - 2\,(5 + 2m)\, x + 3\,(7 + 10m) = 0$$

have (1) equal roots, (2) reciprocal roots ?　　[*C. U. 1936*]

7. If the roots of the equation $x^2 - px + q^2 = 0$ be real, prove that p cannot lie between $-2q$ and $+2q$.

8. For what value of m will the equation

$$\frac{a}{x + a + m} + \frac{b}{x + b + m} = 1$$

have two roots equal in magnitude and opposite in sign ?

9. Prove that the roots of the equation

$$(x - a)(x - b) + (x - b)(x - b) + (x - c)(x - a) = 0$$

are always real and cannot be equal unless $a = b = c$.

10. If the roots of $x^2 + 2rx + pq = 0$ are real and unequal, then those of $x^2 - 2\,(p + q)\, x + (p^2 + q^2 + 2r^2) = 0$ are imaginary and *vice versa*.

11. Show that the values of x obtained from the equations $ax^2 + by^2 = 1$ and $ax + by = 1$ will be equal, if $a - b = 1$.

12. In a quadratic equation of the form $x^2 + px + q = 0$, the absolute-term is misprinted 45 for 48, and the roots are therefore obtained as 5 and 9. Find the roots of the equation correctly printed.

13. Obtain a quadratic equation with rational coefficients one of whose roots is

(i) $\dfrac{1}{3+\sqrt{5}}$.

(ii) $\dfrac{\sqrt{a}}{\sqrt{a}+\sqrt{(a-b)}}$.

(iii) $\dfrac{1}{3+\sqrt{(-2)}}$.

(iv) $\dfrac{p-\sqrt{p^2-4q}}{p+\sqrt{p^2-4q}}$.

14. Find the values of the following expressions

(i) $x^3-12x^2+51x-76$, when $x=4-\sqrt{-3}$.

(ii) $x^4+4x^3+6x^2+4x+9$, when $x=\sqrt{-2}-1$.

(iii) x^3-2x^2-7x+8, when $x=2+\sqrt{3}$.

15. If one root of the equation $x^2-px+q=0$ be twice the other, show that $2p^2=9q$. $\qquad$ [C. U. 1937]

16. If one root of the equation $ax^2+bx+c=0$ is the square of the other, prove that $b^3+a^2c+ac^2=3abc$.

17. Form the quadratic whose roots shall be

(i) m times the roots of $x^2-px+q=0$,

(ii) greater by h than the roots of $x^2-px+q=0$,

(iii) cubes of the roots of $x^2-px+q=0$,

(iv) arithmetic mean and geometric mean of the roots of $x^2-px+q=0$.

18. Find the condition that one root of $ax^2+bx+c=0$ may be

(i) n times the other root,

(ii) nth power of the other root.

19. (i) Find the condition that the roots of the equation $ax^2+bx+c=0$ may be in the ratio $m:n$.

(ii) If the roots of $ax^2 + bx + c = 0$ be in the ratio $p : q$, show that

$$\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} + \sqrt{\frac{c}{a}} = 0.$$

20. (i) If α and β be the roots of $x^2 + px + q = 0$, show that $\alpha\beta^{-1}$ is a root of $qx^2 - (p^2 - 2q)x + q = 0$. [C. U. 1931]

(ii) If α be a root of the equation $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root.

21. Show that the arithmetic mean of the roots of $x^2 - 2ax + b^2 = 0$ is the geometric mean of the roots of $x^2 - 2bx + a^2 = 0$ and *vice versa*.

22. If the ratio of the roots of $ax^2 + bx + c = 0$, be equal to that of the roots of $a_1x^2 + b_1x + c = 0$, then
$$b^2 : b_1^2 :: ac : a_1c_1.$$

23. Determine the values of m for which $3x^2 + 4mx + 2 = 0$ and $2x^2 + 3x - 2 = 0$ may have a common root.

[C. U. 1934]

24. If the equations $x^2 + px + q = 0$ and $x^2 + p'x + q' = 0$ have a common root, show that it must be either
$$\frac{pq' - p'q}{q - q'} \text{ or, } \frac{q - q'}{p' - p}.$$

25. Show that the equations $(q - r)x^2 + (r - p)x + (p - q) = 0$ and $(r - p)x^2 + (p - q)x + (q - r) = 0$ have a common root.

26. If the quadratic $ax^2 + bx + c = 0$ and $bx^2 + cx + a = 0$ have a common root, then either $a + b + c = 0$, or, $a = b = c$.

27. If $a + b + c = 0$, show that each pair of the equations $x^2 + ax + bc = 0$, $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ will have a common root.

28. Prove that if the quadratics $x^2 + px + q = 0$ and $x^2 + qx + p = 0$ have a common root, then either $p = q$ or $p + q + 1 = 0$.

[C. U. 1939]

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29. Prove that if the quadratics $x^2 + bx + ca = 0$ and $x^2 + cx + ab = 0$ have a common root, their other roots will satisfy $x^2 + ax + bc = 0$.

ANSWERS

1. (i) Identity. (ii) Identity. **6.** (i) $2, \frac{1}{2}$. (ii) $-\frac{2}{3}$.

8. 0. **12.** 6, 8. **13.** (i) $4x^2 - 6x + 1 = 0$.

(ii) $bx^2 - 2ax + a = 0$. (iii) $11x^2 - 6x + 1 = 0$.

(iv) $qx^2 - (p^2 - 2q)x + q = 0$. **14.** (i) 0. (ii) 12. (iii) 6.

17. (i) $x^2 - mpx + m^2q = 0$. (ii) $x^2 - (p + 2h)x + (h^2 + ph + q) = 0$.

(iii) $x^2 + (3pq - p^3)x + q^3 = 0$. (iv) $x^2 - (\frac{1}{2}p + \sqrt{q})x + \frac{1}{2}p\sqrt{q} = 0$.

18. (i) $(n+1)^2 ac = nb^2$. (ii) $\left(\dfrac{c}{a}\right)^{\frac{1}{n+1}} + \left(\dfrac{c}{a}\right)^{\frac{n}{n+1}} = \dfrac{b}{a}$.

19. (i) $ac(m+n)^2 = b^2 mn$. **23.** $\frac{7}{4}, -\frac{11}{8}$.

46. Symmetric functions of Roots.

Any expression involving the roots of a quadratic symmetrically, can be easily expressed in terms of the coefficients of the quadratic. The general rule in such a case as "*Express the given expression in terms of the sum and the product of the roots and then write down the values of the sum and the product in terms of the coefficients*". This is illustrated in the examples below.

We shall also illustrate by examples the formation of quadratics whose roots are the symmetric functions of the roots of a given quadratic.

Ex. 1. *If a and β are the roots of the equation $ax^2 + bx + c = 0$, express $a^2 + \beta^2$ in terms of a, b, c.* [*C. U. 1935*]

$$a^2 + \beta^2 = (a + \beta)^2 - 2a\beta = \left(-\frac{b}{a}\right)^2 - 2\frac{c}{a}$$
$$= \frac{b^2}{a^2} - \frac{2c}{a} = \frac{b^2 - 2ac}{a^2}.$$

Ex. 2. *If a and β are the roots of the equation $x^2 + px + q = 0$, find the values of (i) $a\beta^{-2} + \beta a^{-2}$ and (ii) $a^4\beta^7 + \beta^4 a^7$.*

Since $\alpha+\beta=-p$ and $\alpha\beta=q$, we have

(i) $\alpha\beta^{-2}+\beta\alpha^{-2}=\dfrac{\alpha}{\beta^2}+\dfrac{\beta}{\alpha^2}=\dfrac{\alpha^3+\beta^3}{\alpha^2\beta^2}=\dfrac{(\alpha+\beta)^3-3\alpha\beta(\alpha+\beta)}{\alpha^2\beta^2}$

$$=\dfrac{(-p)^2-3q}{q^2}\dfrac{(-p)}{}=\dfrac{3pq-p^3}{q^2}.$$

(ii) $\alpha^4\beta^7+\beta^4\alpha^7=\alpha^4\beta^4\,(\alpha^3+\beta^3)=(\alpha\beta)^4\,\{(\alpha+\beta)^3-3\alpha\beta\,(\alpha+\beta)\}$

$$=q^4\,\{(-p)^3-3q\,(-p)\}$$

$$=pq^4\,(3q-p^2).$$

Ex. 3. *If α and β are the roots of $x^2+px+q=0$, form the equation whose roots are $(\alpha-\beta)^2$ and $(\alpha+\beta)^2$.* [*C. U. 1938*]

Here $\alpha+\beta=-p$ and $\alpha\beta=q$.

Sum of the roots of the required equation

$$=(\alpha-\beta)^2+(\alpha+\beta)^2=2(\alpha^2+\beta^2)$$

$$=2\,\{(\alpha+\beta)^2-2\alpha\beta\}=2(p^2-2q).$$

Product of the roots $=(\alpha-\beta)^2\,(\alpha+\beta)^2=(\alpha+\beta)^2\,\{(\alpha+\beta)^2-4\alpha\beta\}$

$$=p^2\,(p^2-4q)=p^4-4p^2q.$$

Hence, the required equation is

$$x^2-2\,(p^2-2q)\,x+p^4-4p^2q=0.$$

Examplex VIII(B)

1. If α, β are the roots of the equation $ax^2+bx+c=0$, find the values of the following in terms of a, b, c

(i) $\dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$. 　　(ii) $\dfrac{\alpha^2}{\beta}+\dfrac{\beta^2}{\alpha}$. 　　(iii) $\left(\dfrac{\alpha}{\beta}-\dfrac{\beta}{\alpha}\right)^2$.

2. If α, β are the roots of the equation $x^2+px+q=0$, find in terms of p, q the values of the following :

(i) $\alpha^3\beta^{-1}+\beta^3\alpha^{-1}$. 　　　(ii) $\alpha^4+\alpha^2\beta^2+\beta^4$.

(iii) $\alpha^{-3}+\beta^{-3}$. 　　　　(iii) $(1+\alpha+\alpha^2)(1+\beta+\beta^2)$.

(v) $\alpha^2\,(\alpha^2\beta^{-1}-\beta)+\beta^2\,(\beta^2\alpha^{-1}-\alpha)$. 　[*C. U. 1941*]

(vi) $(\alpha+p)^{-4}+(\beta+p)^{-4}$.

3. If α and β are the roots of $ax^2+bx+c=0$, find the equation whose roots are

(i) α^2 and β^2. 　　　　　　　[*C. U. 1933*]

(ii) $\alpha\beta^{-1}$ and $\beta\alpha^{-1}$. 　　　　　[*C. U. 1936*]

(iii) $a^2 + \beta^2$ and $a^{-2} + \beta^{-2}$.

(iv) $a + \beta^{-1}$ and $\beta + a^{-1}$.   [C. U. 1939]

4. If p and q are the roots of the equation $x^2 + 7x + 12 = 0$, find the equation whose roots are $(p+q)^2$ and $(p-q)^2$.

[C. U. 1945]

5. Express the roots of the equation
$$q^2 x^2 - (p^2 - 2q) x + 1 = 0$$
in terms of those of $x^2 + px + q = 0$.

6. If a, β be the roots of $px^2 + qx + q = 0$, show that

$$\sqrt{\frac{a}{\beta}} + \sqrt{\frac{\beta}{a}} + \sqrt{\frac{q}{p}} = 0.$$

7. If a, β are the roots of $x^2 - px + q = 0$, form the equation whose roots are

(i) $ma + n\beta$ and $na + m\beta$. (ii) $a + a^2 \beta^{-1}$ and $\beta + \beta^2 a^{-1}$.

(iii) $2a - \beta$ and $2\beta - a$.

8. If a, β and γ, δ be the roots of $x^2 - px + q = 0$ and $x^2 - p'x + q' = 0$, find the value of $(a - \gamma)(\beta - \delta) + (\beta - \gamma)(a - \delta)$.

9. If a, β and a', β' be the roots of $x^2 - px + q = 0$ and $x^2 - p'x + q' = 0$ respectively, find the value of
$$(a - a')^2 + (a - \beta')^2 + (\beta - a')^2 + (\beta - \beta')^2.$$

10. If a, β and λ, δ be the roots of $x^2 + px - r = 0$ and $x^2 + px + r = 0$ respectively, prove that
$$(a - \gamma)(a - \delta) = (\beta - \gamma)(\beta - \delta).$$

11. If a, β and γ, δ are the roots of $x^2 - 2ax + b^2 = 0$ and $x^2 - 2cx + d^2 = 0$ and if $a\delta = \beta\gamma$, prove that $a^2 d^2 = b^2 c^2$.

12. If the roots of $ax^2 + bx + c = 0$ differ from those of $a'x^2 + b'x + c' = 0$ by a constant, show that
$$\frac{b^2 - 4ac}{a^2} = \frac{b'^2 - 4a'c'}{a'^2}.$$

13. Form the quadratic whose roots are the reciprocals of the roots of $ax^2 + bx + c = 0$.   [C. U. 1944]

14. If the roots of $ax^2 + bx + c = 0$ are the reciprocals of those of $a'x^2 + b'x + c' = 0$, show that $a : b : c : : c' : b' : a'$.

15. If the roots of $x^2 + 2px + q = 0$ and $x^2 + 2qx + p = 0$ differ by a constant, show that $p + q + 1 = 0$. [*C. U. 1944*]

16. If one root of the equation $x^2 + ax + b = 0$ be a root of the equation $x^2 + cx + d = 0$, its other root is a root of the equation, $x^2 + (2a - c)x + (a^2 - ac + d) = 0$.

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1. (i) $\dfrac{b^2 - 2ac}{c^2}$. 　(ii) $\dfrac{b(3ac - b^2)}{a^2 c}$. 　(iii) $\dfrac{b^2(b^2 - 4ac)}{a^2 c^2}$.

2. (i) $\dfrac{p^4 - 4p^2 q + 2q^2}{q}$. 　(ii) $(p^2 - q)(p^2 - 3q)$.

(iii) $\dfrac{3pq - p^3}{q^3}$. 　(iv) $1 - p + p^2 - q - pq + q^2$.

(v) $\dfrac{p(p^2 - 4q)(q - p^2)}{q}$. 　(vi) $\dfrac{p^4 - 4p^2 q + 2q^2}{q^4}$.

3. (i) $a^2 x^2 - (b^2 - 2ac)x + c^2 = 0$. 　(ii) $ac(x + 1)^2 = b^2 x$.

(iii) $a^2 c^2 x^2 - (a^2 b^2 + b^2 c^2 - 2a^3 c - 2ac^3)x + (b^2 - 2ac)^2 = 0$.

(iv) $acx^2 + b(a + c)x + (a + c)^2 = 0$. 　**4.** $x^2 - 50x + 49 = 0$.

5. α^{-2}, β^{-2} ; α, β being the roots of the later equation.

7. (i) $x^2 - p(m + n)x + mnp^2 + q(m - n)^2 = 0$.

(ii) $qx^2 - p(p^2 - 2q)x + p^2 q = 0$.

(iii) $x^2 - px + 9q - 2p^2 = 0$. 　**8.** $2(q + q') - pp'$.

9. $2(p^2 - 2q + p'^2 - 2q' - pp')$. 　**13.** $cx^2 + bx + a = 0$.

47. Factors of the Quadratic expression $ax^2 + bx + c$.

Let α, β be the roots of the quadratic equation

$$ax^2 + bx + c = 0,$$

so that $\quad \alpha + \beta = -\dfrac{b}{a}$ and $\alpha\beta = \dfrac{c}{a}$.

Now, $ax^2 + bx + c = a\left(x^2 + \dfrac{b}{a}x + \dfrac{c}{a}\right)$

$$= a\{x^2 - (\alpha + \beta)x + \alpha\beta\}$$

$$= a(x - \alpha)(x - \beta).$$

Note. The student should note carefully the distinction between *a quadratic equation* and *a quadratic expression*. In the quadratic equation $ax^2 + bx + c = 0$, x can have only *two* values but in the quadratic expression $ax^2 + bx + c$, x can have *any* value.

Cor. 1. If a, β be the roots of $x^2 + px + q = 0$, then
$$x^2 + px + q = (x - a)(x - \beta).$$

Cor. 2. It follows from above that $ax^2 + bx + c$ is divisible by $x - a$ *if a is a root of $ax^2 + bx + c = 0$.*

We have seen another method of proof of this important result in Art. 40. We give below a third method.

By actual division we see

$$x - a \overline{\smash{)}\, ax^2 + bx + c} \left(ax + (aa + b) \right.$$
$$\underline{ax^2 - aax}$$
$$x(aa + b) + c$$
$$\underline{x(aa + b) - aa^2 - ba}$$
$$aa^2 + ba + c$$

Thus, we see that if $ax^2 + bx + c$ be divisible by $x - a$, we must have $aa^2 + ba + c = 0$, *i.e.*, a must be a root of $ax^2 + bx + c = 0$.

Note 2. It is evident from above that the nature of the factor of a quadratic expression depends upon the nature of the corresponding quadratic equation. Hence the factors will be (i) *rational*, if $b^2 - 4ac$ is positive and a perfect square and a, b, c are rational ; (ii) *real and irrational*, if $b^2 - 4ac$ is positive but not a perfect square ; (ii) *imaginary*, if $b^2 - 4ac$ is negative : (iv) *real and equal*, if $b^2 - 4ac = 0$. When the factors are equal, then $ax^2 + bx + c = a(x - a)^2 = \{\sqrt{a}(x - a)\}^2$. Thus $ax^2 + bx + c$ is a *perfect* square, if $b^2 = 4ac$ and $a > 0$.

48. Condition for a common linear factor.

Since a factor of the expression $ax^2 + bx + c$ corresponds to a root of the equation $ax^2 + bx + c = 0$, it follows that if two quadratic expressions $ax^2 + bx + c$ and $a'x^2 + b'x + c'$ have a common linear factor, the two corresponding equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ have a common root, the condition for which is [See *Art. 46, Ex. 10.*]
$$(bc' - b'c)(ab' - a'b) = (ca' - c'a)^2.$$

This is therefore the required condition that the above two quadratic expressions have a common linear factor.

49. Factors of the general Quadratic expression in x and y.

The general quadratic expression in x and y can be resolved into two linear factors under certain conditions.

Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ be the general quadratic expression in x and y.

Consider the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

or, $$ax^2 + 2x\,(hy + g) + (by^2 + 2fy + c) = 0$$

which, treated as a quadratic in x, gives

$$x = \frac{-(hy + g) \pm \sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}}{a}.$$

$\therefore$ The exp. $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

$$= a\left\{x + \frac{hy + g - \sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}}{a}\right\}$$

$$\times \left\{x + \frac{hy + g + \sqrt{(hy + g)^2 - a(by^2 + 2fy + c)}}{a}\right\}.$$

Thus, the two factors will be both linear, if

$$(hy + g)^2 - a\,(by^2 + 2fy + c) \text{ is a perfect square,}$$

i.e.,   if $y^2\,(h^2 - ab) + 2y\,(gh - af) + (g^2 - ac)$

is a perfect square, which requires that

$$4\,(gh - af)^2 = 4\,(h^2 - ab)(g^2 - ac),$$

or, $$g^2h^2 - 2afgh + a^2f^2 = h^2g^2 - h^2ac - abg^2 + a^2bc,$$

i.e., $$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

This is the required condition, and the expression on the left is called the *discriminant* of the general quadratic expression.

50.   The sign of the expression $ax^2 + bx + c$.

The sign of the expression $ax^2 + bx + c$ depends on the nature of the roots of the equation, $ax^2 + bx + c = 0$.

If a, β are the roots of the equation, then

$$ax^2 + bx + c = a\,(x - a)(x - \beta).$$

Three different cases are to be considered.

(i) The roots *real* an *equal*, $(a = \beta)$:

$\therefore \quad ax^2 + bx + c = a\,(x - a)^2 = a \times$ a positive quantity ;

$\therefore \quad ax^2 + bx + c$ has the *same sign* as a.

(ii) The roots *imaginary* :

Let $a = m + in$ and $\beta = m - in$.

$$\begin{aligned}
\therefore \quad ax^2 + bx + c &= a\,\{x - (m + in)\}\{x - (m - in)\} \\
&= a\,\{(x - m) - in\}\{(x - m) + in\} \\
&= a\,\{(x - m)^2 + n^2\},\ \text{since}\ i^2 = -1 \\
&= a \times \text{a positive quantity} :
\end{aligned}$$

$\therefore \quad ax^2 + bx + c$ has the *same sign* as a.

(iii) The roots *real* and *unequal* :

Suppose x *lies between* a and β.

Now, if $a < x < \beta$, then $x - a$ is positive and $x - \beta$ is negative.

But if $\beta < x < a$, then $x - a$ is negative and $x - \beta$ is positive.

$\therefore$ In either case, $(x - a)(x - \beta)$ is negative and hence $a\,(x - a)(x - \beta)$ is positive or negative, according as a is negative or positive.

$\therefore \quad ax^2 + bx + c$ *has a sign opposite to that of* a.

Next suppose x *does not lie between* a *and* β.

Now, if x be greater than both a and β, then $x - a$ and $x - \beta$ are both positive.

But if x be less than both a and β, then $x - a$ and $x - \beta$ are both negative.

$\therefore$ In either case, $(x - a)(x - \beta)$ is positive and hence $a(x - a)(x - \beta)$ is positive, or negative, according as a is positive or negative.

$\therefore$ $ax^2 + bx + c$ *has the same sign as* a.

Thus, *for all real values of* x, *the expression* $ax^2 + bx + c$ *has the same sign as* a, *except when the roots of the equation* $ax^2 + bx + c = 0$ *are real and unequal and* x *lies between them.*

51. Magnitude of the expression ax² + bx + c.

Let $ax^2 + bx + c = y$; then $ax^2 + bx + c - y = 0$.

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4a(c - y)}}{2a}.$$

Since x is real, the expression under the radical sign must not be negative [See *Art. 41*].

$$\therefore \qquad b^2 - 4ac + 4ay \not< 0$$

$$i.e., \quad 4a\left\{y - \frac{4ac - b^2}{4a}\right\} \not< 0 \qquad \cdots \qquad \cdots \quad (1)$$

$$i.e., \quad \left\{y - \frac{4ac - b^2}{4a}\right\} \not< 0, \textit{ if } a \textit{ is positive} \qquad \cdots \quad (2)$$

$$\text{and} \qquad \not> 0, \textit{ if } a \textit{ is negative} \qquad \cdots \quad (3)$$

$\therefore$ From (2), it follows that if a *is positive*, for all real values of x, the value of y, *i.e.*, of $ax^2 + bx + c$ is $>$ or $= \dfrac{4ac - b^2}{4a}$ and hence in this case the *least or minimum* value of y is $\dfrac{4ac - b^2}{4a}$;

and from (3), it follows that if *a is negative*, for all real values of x, the value of y, *i.e.*, of $ax^2 + bx + c$ is < or $= \dfrac{4ac - b^2}{4a}$ and hence in this case the *greatest* or *maximum* value of y is $\dfrac{4ac - b^2}{4a}$.

Note 1. It is evident from above that when the expression is either maximum or minimum $x = -\dfrac{b}{2a}$.

Note 2. The above result can also be established in another way. This is illustrated in the examples below.

52. Illustrative Examples.

Ex. 1. *Prove that for real values of x, the expression $3x^2 - 6x + 8$ can never be less than 5.* [*C. U. 1935*]

$$3x^2 - 6x + 8 = 3(x^2 - 2x + 1) + 5 = 3(x-1)^2 + 5.$$

Since, for all real values of x, $(x-1)^2$ is never negative,

∴ $3x^2 - 6x + 8$ can never be less than 5.

Ex. 2. *If x be real, show that the least value of $4x^2 - 4x + 1$ is zero and the corresponding value of x is $\frac{1}{2}$.* [*C. U. 1937*]

$4x^2 - 4x + 1 = (2x-1)^2$. ∴ It is either zero or positive for real values of x, and hence its least value is 0 ; and the corresponding value of x is given by $2x - 1 = 0$, *i.e.*, $x = \frac{1}{2}$.

Ex. 3. *If x is real, prove that $\dfrac{x^2 + 34x - 71}{x^2 + 2x - 7}$ can have no value between 5 and 9.* [*C. U. 1938*]

Let $y = \dfrac{x^2 - 34x - 71}{x^2 + 2x - 7}$.

Then $x^2(y-1) + 2(y-17)x - (7y - 71) = 0$,

∴ $x = \dfrac{-2(y-17) \pm \sqrt{4(y-17)^2 + 4(y-1)(7y-71)}}{2(y-1)}$

Since, x is real, the expression under the radical sign, *i.e.*,

$$4 \{(y-17)^2 + (y-1)(7y-71)\} \text{ must not be negative}$$

i.e., $4 \{8 (y^2 - 14y + 45)\}$ ········

i.e., $32 \{(y-5)(y-9)\}$ ········

∴ y can have no value between 5 and 9, for when $y > 5$ but < 9, one of the factors, *viz.*, $(y-9)$, becomes negative and hence the expression becomes negative.

Examples VIII(C)

1. (i) If x is real and expression

$$x^2 + (b - 2a) x + a (a - b),$$

is positive, show that x can never lie between a and $a - b$.

(ii) If x is real, show that $4x + 7 - 3x^2$ will never be greater than $\frac{25}{3}$.

(iii) Show that $2x^2 + 5x + 6$ is positive for all real values of x.

(iv) Prove that $10x^2 - 17x + 3$ is positive for all real values of x except when x lies between $\frac{1}{5}$ and $\frac{3}{2}$.

2. (i) Show that $(x-1)(x-2)(x-3)(x-4) + 5$ is positive for real values of x.

(ii) Divide the number 10 into two parts such that the sum of their squares is the least possible.

3. Show that $\dfrac{x^2 - 3x + 4}{x^2 + 3x + 4}$ lies between 7 and $\frac{1}{7}$, if x be real. [*C. U. 1940*]

4. If x be real, prove that $\dfrac{2x^2 - 2x + 4}{x^2 - 4x + 3}$ cannot lie between 1 and -7. [*C. U. 1944*]

5. Prove that $\dfrac{2x^2 + 5x + 2}{3x^2 + 4x + 1}$ can have any real value if x be real.

6. If x be real and a have any value between 1 and 3, show that $\dfrac{ax^2 + x - 2}{a + x - 2x^2}$ can have any real value.

7. Find the limits between which $\dfrac{x^2 - x + 1}{x^2 + x + 1}$ must lie for real values of x.

8. (a) If x is real, find
      (i) the greatest value of $(x + 1)(3 - x)$.
      (ii) the least value of $(x + 2)(x + 3)$.

    (b) (i) Find the maximum value of $-3x^2 + 6x - 1$ and the value of x for which the expression is maximum.

      (ii) Find the values of a which make $x^2 - ax + 1 - 2a^2$ always positive for real values of x.

9. Show that the greatest and least values of
$$\frac{6x^2 - 22x + 21}{5x^2 - 18x + 17} \qquad\qquad [\,C.\,U.\,1944\,]$$
for all real values of x are $\frac{5}{4}$ and 1 corresponding to the values 1 and 2 respectively of x.

10. Prove that the expression $x^2 + 2bx + c$ is greater than, equal to, or less than $(x + b)^2$, according as the roots of the equation $x^2 + 2bx + c = 0$ are imaginary, equal, or real.

11. If x and y are two real quantities connected by the equation $x^2 + 12xy + 4y^2 - 26x - 44y + 89 = 0$, then x cannot lie between 4 and 1 and y between $\frac{5}{3}$ and 1.

12. Show that if the quadratic expression
$$x^2 + 2\,(a + b + c)\,x + 3\,(ab + bc + ca)$$
be a perfect square, then $a = b = c$.

13. Find the condition that the expression
$$ax^2 + 2hxy + by^2$$
may have two factors of the forms $y - mx$ and $my + x$.

14. Determine the values of a for which
$$4x^2 - 12xy + 4y^2 + 4x - 2(3 - a)y - 4$$
is resolvable into linear factors.

15. Show that if $ax^2 + by^2 + cz^2 + 2ayz + 2bzx + 2cxy$ is resolvable into linear factors, then $a^3 + b^3 + c^3 - 3abc = 0$.

16. If x is real, show that $2x - x^2 - 3$ can have all values between $-\infty$ and -2 and none between -2 and $+\infty$.

17. If $axy + bx + cy + d$ can be resolved into factors, show that $ad = bc$.

18. Find the condition that
$$ax^2 + 2hxy + by^2 \text{ and } a'x^2 + 2h'xy + b'y^2$$
may have factors of the forms $y - mx$ and $my + x$ respectively.

19. Factorise $6x^2 - 19xy + 15y^2 + 23x - 36y + 21$.

20. If $p > 1$, then for real values of x, the expression $\dfrac{x^2 - 2x + p^2}{x^2 + 2x + p^2}$ lies between $\dfrac{p-1}{p+1}$ and $\dfrac{p+1}{p-1}$.

21. Prove that if x is real, the expression $\dfrac{(x-a)(x-c)}{(x-b)}$ is capable of assuming all values if a, b, c are either in ascending or in descending order of magnitude.

22. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ is a perfect square, show that $f = \pm \sqrt{(bc)}$, $g = \pm \sqrt{(ca)}$ and $h = \pm \sqrt{(ab)}$, provided that the negative sign does not precede an odd number of the radicals.

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2. (ii) 5, 5.   **7.** 3. $\frac{1}{3}$.   **8.** (a) (i) 4.   (ii) $-\frac{1}{4}$.

(b) (i) 2 ; $x = 1$.   (ii) a lies between $-\frac{2}{3}$ and $+\frac{2}{3}$.

13. $a + b = 0$.   **14.** ± 5.

Examples VIII(D)

[Additional Examples on Chapter VIII]

1. If one root of the equation $ax^2 + bx + c = 0$ be four times the other, show that $4b^2 = 25ac$.   [*C. U. 1940*]

2. If the difference of the roots of the equation $x^2 - px + q = 0$ be the same as that of the equation $x^2 - qx + p = 0$, show that $p + q + 4 = 0$, unless $p = q$.

[*C. U. 1941*]

3. If one of the roots of $x^2 + px + q = 0$ is the square of the other, show that $p^3 - q(3p - 1) + q^2 = 0$.   [*C.U. 1943*]

4. If a, β are the roots of $x^2 + px + 1 = 0$ and γ, δ are the roots of $x^2 + qx + 1 = 0$, show that

$$(a - \gamma)(\beta - \gamma)(a + \delta)(\beta + \delta) = q^2 - p^2.$$

5. If each pair of the three equations

$$x^2 - p_1 x + q_1 = 0, \ x^2 - p_2 x + q_2 = 0, \ x^2 - p_3 x + q_3 = 0$$

have a common root (not common to all three), prove that

$$p_1{}^2 + p_2{}^2 + p_3{}^2 + 4(q_1 + q_2 + q_3) = 2(p_2 p_3 + p_3 p_1 + p_1 p_2).$$

6. Show that the ratio r of one root of the equation $ax^2 + bx + c = 0$ to the other, is given by the equation

$$acr^2 + (2ac - b^2)r + ac = 0.$$

7. If the equation $ax^2 + bx + c = 0$ is not altered when each of its coefficients is increased by the same quantity, show that $x^3 = 1$.

8. If the roots of the equation $ax^2 + 2bx + c = 0$ be a and β and those of the equation $Ax^2 + 2Bx + C = 0$ be $a + \delta$ and $\beta + \delta$, prove that $\dfrac{b^2 - ac}{B^2 - AC} = \left(\dfrac{a}{A}\right)^2$.

9. If α and β be the roots of the equation $x^2 + 2ax + b = 0$, form a quadratic with rational coefficients, one of whose roots is $\alpha + \beta + \sqrt{(\alpha^2 + \beta^2)}$.

10. Find λ so that the values of x given by

$$\frac{\lambda}{2x} = \frac{a}{x + c} + \frac{b}{x - c}$$

may be equal. If λ_1, λ_2 are the two values of λ and x_1, x_2 the corresponding values of x, show that

$$\lambda_1 \lambda_2 = (a - b)^2 \text{ and } x_1 x_2 = c^2.$$

11. If the equation $ax^2 + 2bx + c = 0$, $a'x^2 + 2b'x + c' = 0$ have a common root, prove that the equation

$$(b^2 - ac)x^2 + (2bb' - ac' - a'c)x + (b'^2 - a'c') = 0$$

has equal roots.

12. Find the quadratic equation one of whose roots is

(i) $\dfrac{2ab}{(a + b) - \sqrt{(a^2 + b^2)}}$. (ii) $\dfrac{a^2 + b^2}{(a - b) + \sqrt{(-2ab)}}$.

13. If $\dfrac{(x - 5)(x^2 - 2x + 1)}{(x - 7)(x^2 + 2x + 3)}$ is positive for real values of x, show that x cannot lie between 5 and 7.

14. If $ay - bx = c\sqrt{(x - a)^2 + (y - b)^2}$, show that no real values of x and y will satisfy the equation unless

$$c^2 < (a^2 + b^2).$$

15. If $x - a$ is a factor of $a_1 x^2 + 2b_1 x + c_1$ and $x + a$, a factor of $a_2 x^2 + 2b_2 x + c_2$, prove that

$$(a_1 c_2 - c_1 a_2)^2 + 4(a_1 b_2 + a_2 b_1)(b_1 c_2 + b_2 c_1) = 0.$$

16. If, α, β are the roots of $ax^2 + bx + c = 0$, find the values of

(i) $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$;

(ii) $(a\alpha + b)^{-3} + (a\beta + b)^{-3}$. [C. U. 1943]

17. Show that the expression $\dfrac{(ax-b)(b'x-a')}{(bx-a)(a'x-b')}$ will be capable of all values when x is real if $a^2 - b^2$ and $a'^2 - b'^2$ have the same sign.

18. If $a \pm \sqrt{\beta}$ be the roots of $x^2 + px + q = 0$, prove that $\dfrac{1}{a} \pm \dfrac{1}{\sqrt{\beta}}$ will be the roots of

$$(p^2 - 4q)(p^2x^2 + 4px) - 16q = 0.$$

19. The difference of the roots of a quadratic equation is a and the quotient obtained by dividing the greater root by the smaller is b. Find the equation.

20. Show that the expression
$$\frac{(x^2 - 4)(x^2 + 3x + 2)(x^2 - x - 2) + 10}{x^2 + 5x + 7}$$
is positive for all real values of x.

ANSWERS

9. $x^2 + 4ax + 2b = 0.$ 10. $a + b \pm 2\sqrt{(ab)}.$

12. (i) $x^2 - 2(a+b)x + 2ab = 0.$ (ii) $x^2 - 2x(a-b) + a^2 + b^2 = 0.$

16. (i) $\dfrac{b^2 - 2ac}{a^2c^2}.$ (ii) $\dfrac{b(b^2 - 3ac)}{a^3c^3}.$

19. $(b-1)^2x^2 - a(b^2-1)x + a^2b = 0.$

SUPPLEMENT TO CHAPTER VII

THE CONCEPT OF SIMPLE ALGEBRAIC FUNCTIONS

52'1. Polynomials.

Any algebraic expression containing only one variable x is called an algebraic function of x.

A function of x is generally denoted by $f(x)$. It may also be denoted by other symbols, such as, $g(x)$, $\phi(x)$, $\psi(x)$, $F(x)$, etc.

A function of x of the type,

$$f(x) \equiv ax^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

is called a **polynomial,** a_0, a_1, a_2,......, a_{n-1}, a_n being independent of x.

Note. $f(x) \equiv a_0$, where a_0 is a constant, should also be considered as a function of x, retaining the constant value a_0 for all values of x.

$f(x) \equiv a_0 x + a_1$ is called a *linear* function of x.

$f(x) \equiv a_0 x^2 + 2a_1 x + a_2$ is called a *quadratic* function of x.

$f(x) \equiv a_0 x^3 + 3a_1 x^2 + 3a_2 x + a_3$ is calld a *cubic* function of x.

$f(x) \equiv a_0 x^4 + 4a_1 x^3 + 6a_2 x^2 + 4a_3 x + a_4$ is called a *biquadratic* or *quartic* function of x ; and so on.

52'2. Rational Algebraic Functions.

The function $\dfrac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomials in x, is called a **rational algebraic functions of x.**

Note. Since "Division by zero (*i.e.* $a/0$, $0/0$) is meaningless", the function $\dfrac{f(x)}{g(x)}$ is *undefined* at the zeros of $g(x)$.

8

Ex. 6. Find the greatest and the least values of $f(x) \equiv \dfrac{x^2 - x + 1}{x^2 + x + 1}$, for values of x. Also find the corresponding values of x.

[*Greatest* $= 3$, *for* $x = -1$.   *Least* $= \frac{1}{3}$, *for* $x = 1$.]

Ex. 7. Show that, when $(ax + b)$ is divided by $(x - a)$, the remainder is $(aa + b)$.

Ex. 8. Show that, when $(ax^2 + bx + c)$ is divided by $(x - a)$, the remainder is $(aa^2 + ba + c)$.

Ex. 9. Find the remainder when $(3x^3 - 7x^2 + 4)$ is divided by $(x - 2)$, without performing actual division.   [0]

Ex. 10. If p, q, r are *rational*, and $f(x) \equiv px^2 + qx + r$, where $f(2) = 0$, prove that the roots of $f(x) = 0$ are *rational*.

Ex. 11. Find the least value of $f(x) \equiv \left(x + \dfrac{1}{x}\right)$, for positive values of x, and the greatest value of $f(x)$, for negative values of x.

[2 ; -2]

Ex. 12. If $\phi(x) \equiv px^2 + qx + r$, $\psi(x) \equiv lx^2 + mx + n$, prove the following :

(i) $\phi(x) = 0$ and $\psi(x) = 0$ have the same roots, when $p/l = q/m = r/n$;

(ii) $\phi(x) = 0$ and $\psi(x) = 0$ have the roots of one reciprocal to those of the other, when $p/n = q/m = r/l$;

(iii) $\phi(x) = 0$ and $\psi(x) = 0$ have their roots proportional when $q^2/pr = m^2/ln$.

Ex. 13. For the function, $f(x, y) \equiv (x - y)^2 - 4(x + y) + 8$, show that $f(x, y) = f(y, x)$.

PERMUTATIONS AND COMBINATIONS
SEC. A. PERMUTATIONS

53. Permutations defined.

The different *arrangements*, that can be made out of a given set of things, by taking some or all of them are called their *permutations*.

Thus, *ab, ba* are two different permutations of the two letters *a, b.* The permutations of three letters *a, b, c taken two at a time* are *ab, ba, ac, ca, bc, cb* and the permutations of these three letters *a, b, c* taken *all at a time* are obviously *abc, acb, bca, bac, cab, cba.*

The number of permutations of n different things taken r at a time is usually denoted by the symbol ^nP_r or $_nP_r$.

54. Permutations of things all different.

To find the number of permutations of n different things taken r at a time $(r \leqslant n)$.

The number of permutations of n different things taken r at a time, is the same as the number of different ways in which r places can be filled up by the n things.

The first place can be filled up in n ways, for any one of the n things can be put in it. When the first place has been filled up in *any one* of these n ways, the second place can be filled up in $(n-1)$ ways, for any one of the remaining $(n-1)$ things can be put in it. Since each way of filling up the the first place can be associated with each way of

filling up the second place, the first two places can be filled up in $n(n-1)$ ways.

When the first two places have been filled up in *any one* of these $n(n-1)$ ways, the third place can be filled up in $(n-2)$ ways, for there are now $(n-2)$ things left to fill up the third place. Each of these $(n-2)$ ways can be associated with each of the $n(n-1)$ arrangements of the first two places. Hence the first three places can be filled up in $n(n-1)(n-2)$ ways.

Proceeding in this way and noticing that at any stage the number of factors is the same as the number of places filled up, the total number of ways in which r places can be filled up

$$= n(n-1)(n-2)(n-3)\cdots\cdots \text{ to } r \text{ factors}$$
$$= n(n-1)(n-2)\cdots\cdots\{n-(r-1)\}$$
$$= n(n-1)(n-2)\cdots\cdots(n-r+1).$$

Hence, $^n P_r = n(n-1)(n-2)\cdots\cdots(n-r+1)$ $\cdots$ (1)

Cor. 1. Putting $r=n$ in the formula (1), we have the *number of permutations of n different things taken all at a time,*

i.e., $^n P_n = n(n-1)(n-2)\cdots\cdots \text{ to } n \text{ factors}$
$$= n(n-1)(n-2)\cdots\cdots 3.\,2.\,1.$$

∴ $^n P_n = \underline{|n}.$ $\cdots$ $\cdots$ $\cdots$ (2)

Obs. The product $1.\,2.\,3\ldots\ldots(n-2).(n-1).\,n,$ *i.e.*, of continued product of the first n natural numbers (*i.e.*, of n consecutive integers beginning with 1 and ending with n) is usually denoted by the symbol $\underline{|n}$ or $n!$ and either symbol is read as *factorial n.*

Thus, $\underline{|5}$ or $5! = 1.2.3.4.5 = 120.$

Cor. 2. $^nP_r = n\,(n-1)(n-2)\cdots\cdots(n-r+1)$

$$= \frac{n\,(n-1)(n-2)\cdots\cdots(n-r+1)\;\underline{n-r}}{\underline{n-r}}$$

$\therefore$ $^nP_r = \dfrac{n\,!}{\underline{n-r}}$ $\cdots$ $\cdot$ $\cdots$ (3)

Note. It is evident that ^nP_r is greatest when $r=n$ or $n-1$; for obviously $^nP_n = {}^nP_{n-1}$.

55. Permutations of things not all different.

To find the number of permutations of n things taken all together, when the things are not all different.

Let the n things be represented by n letters and suppose p of them are a's, q of them are b's, r of them c's and the rest all different.

Let x be the required number of permutations.

If the p a's are changed into p letters different from each other and from the rest, then by changing the arrangement of the p new letters only among themselves, without altering the position of any other letter, each permutation would give rise to $p\,!$ different permutations; hence if this change were made in each of the x permutations, the number of permutations would become $x \times p\,!$.

Similarly, if in any one of these new permutations q b's are changed into q letters different from each other and from the rest, we should obtain from each permutation $q\,!$ permutations by changing the order of these q letters only. Hence the total number of permutation would now become $x \times p\,! \times q\,!$

Again, if the r c's be changed into r letters different from each other and from the rest, the total number of permutations would similarly become $x \times q\,! \times p\,! \times r\,!$.

But now, there are n different things and the permutations of n different things taken all at a time is n !

$\therefore \quad x \times p! \times q! \times r! = n!$.

$\therefore$ x (*i.e.*, **the required number of permutations**)

$$\frac{n!}{p! \, q! \, r!}.$$

Note. The above method is perfectly general and is applicable when more than three kinds of letter are repeated.

56. Permutations involving repetitions.

The number of permutations of n different things taken r at a time, when each thing may be repeated up to r times in any arrangement, is n^r.

Let the n things be represented by n letters.

Evidently, the number of permutations in this case is the same as the number of ways in which r places can be filled up by these n different letters, each letter being repeated once, twice, thrice,... up to r times.

The first place can be filled up in n ways, since any one of the n letters can be put there ; when the first place has been filled up in any one way, the second place can also be filled up in n ways, since the same letter can be used again. Thus, the first two places can be filled up in $n \times n$ or n^2 ways. Similarly, the first three places can be filled up in n^3 ways.

Proceeding in this way and noticing that at any stage the index of n is the same as the number of places filled up, the number of ways in which r places can be filled up is evidently equal to n^r.

Thus, the required number of permutations $= n^r$.

57.   Restricted and Relative positions in Permutations.

There are some problems on permutations in which the positions of things are restricted or assigned ; for example, some things are always to be kept together or some things are to be placed in definite positions etc. ; also some problems occur in which it is required to find the positions of the things relatively to themselves instead of their absolute changes of positions.   These are illustrated in the examples below.

58.   Illustrative Examples.

Ex. 1.   *Three persons enter a room where there are seven vacant chairs in a line ; find without assuming the formula in how many ways they can take their seats.*

The first person can occupy any one of the 7 chairs ; then the second, any one of the remaining 6.   Hence the first two persons can occupy the chairs in 7×6 ways. Corresponding to each way of occupying two chairs by the first two persons, the third person can occupy any one of the remaining 5.

$\therefore$ The total no. of ways reqd. $= 7 \times 6 \times 5 = 210$.

Ex. 2.   *If $^nP_5 : {}^nP_3 = 2 : 1$, find n.*

We have $^nP_5 = 2.{}^nP_3$.

$\therefore$ $n(n-1)(n-2)(n-3)(n-4) = 2n(n-1)(n-2)$.

Cancelling the common factors $n(n-1)(n-2)$ from both sides (since $n = 0, 1, 2$ cannot obviously by the solutions of the equations, the permutations of 1 or 2 things taken 3 or 5 at a time having no meaning), we have

$$(n-3)(n-4) = 2, \quad \text{or}, \quad n^2 - 7n + 10 = 0,$$

or, $(n-2)(n-5) = 0.$ $\therefore$ $n = 2$, or, 5.

Rejecting the inadmissible value 2, we get $n = 5$.

Ex. 3. *Prove that* $\lfloor 2n = \lfloor n.2^n \{1.3.5\ldots\ldots(2n-1)\}$.

$$\lfloor 2n = 1.2.3.4.5\cdots\cdots 2n$$

$$= \{1.3.5\cdots\cdots(2n-1)\} \times (2.4.6\cdots\cdots 2n)$$

$$= \{1.3.5\cdots\cdots(2n-1)\} \times \{(2.1) \cdot (2.2) \cdot (2.3)\ldots\ldots(2.n)\}$$

$$= \{1.3.5\cdots\cdots(2n-1)\} \times 2^n (1.2.3\cdots\cdots n)$$

$$= \lfloor n.2^n \{1.3.5\cdots\cdots(2n-1)\}.$$

Ex. 4. *In how many ways can the letters of the word* laughter *be arranged so that the vowels may never be separated ?*

Considering the 3 vowels as *one*, we have only 6 things l, g, h, t, r, $(a\ u\ e)$ to arrange. Hence the number of arrangement of the letters in which the vowels stand together in the order $(a\ u\ e) = {}^6P_6 = \lfloor 6 = 720$.

Again, the vowels being 3 in number can be arranged among themselves in $\lfloor 3$ or 6 ways.

∴ the total no. of arrangement reqd. $= 720 \times 6 = 4320$.

Ex. 5. *In how many ways can 5 boys form a ring ?*

Let A, B, C, D, E be the 5 boys ; let us make one of them say A fixed and then find the no. of arrangement of the remaining 4 taken all together. This is equal to ${}^4P_4$ *i.e.*, $\lfloor 4$ or 24. Hence the reqd. no. of ways in which 5 boys can form a ring$= 24$.

Note 1. The student should note carefully the distinction between the *arrangements in a row* and the *arrangements in a circle ;* in a circle there will be no "ends" to a set. Thus, the no. of arrangements of n different things (in a row) taken *all* together is $\lfloor n$, whereas the no. of arrangements of n different things (in a circle) taken *all* together is $\lfloor (n-1)$. So in some text-books these two kinds of arrangements are distinguished as *linear* permutations and *circular* permutations.

Note 2. In the above arrangements, *clockwise* and *counter-clockwise* orders are distinguished ; if this distinction is not made, the answer would be $\frac{1}{2} \cdot 24 = 12$. Thus, if 5 beads of unlike colours are threaded on a ring, there are only $\frac{1}{2} \lfloor 4$ or 12 different designs ; for if the ring containing beads in any order, be turned over on its other side, the clockwise and counter-clockwise arrangements become identical.

Note 3. If the question were this :—"In how many ways can 5 persons be seated at a round table ?" The answer would be $^5P_5$ *i.e.*, $\lfloor 5$; for in this case, the positions of the persons with *respect to the table and not with respect to each other* are required.

Ex. 6. *In how many ways can the letters of the words* Mississippi *be arranged ?*

There are 11 letters of which *i* occurs 4 times, *s* occurs 4 times and *p* occurs 2 times.

Hence the reqd. no. of ways,

$$= \frac{\lfloor 11}{\lfloor 4 \lfloor 4 \lfloor 2} \qquad [\textit{ by Art. } 55\ ]$$

$$= \frac{1.2.3.4.5.6.7.8.9.10.11}{1.2.3.4.1.2.3.4.1.2} = 34650.$$

Ex. 7. *In how many ways can the letters of the word* balloon *be arranged, so that the two l's do not come together ?*

Since there are 2 *l's* and 2 *o's*, therefore the no. of arrangements of letters of *balloons* subject to no restriction, is by Art. 55,

$$\frac{\lfloor 7}{\lfloor 2 \lfloor 2} = 1260.$$

The no. of arrangements in which the two *l's* come together is obtained by regarding the two *l's* as forming a single letter and is therefore $= \dfrac{\lfloor 6}{\lfloor 2} = 360.$

Hence the no. of arrangements in which the two *l's* do not come together $= 1260 - 360 = 900.$

Ex. 8. *A servant has to post 5 letters and three are 4 letter-boxes in the locality ; in how many ways can he post the letters ?*

Each letter may be posted in 4 ways, since there are 4 letter-boxes. Again since there are 5 letters, the total no. of ways reqd. $= 4^5 = 1024.$

Ex. 9. *Prove by general reasoning that*
$$^nP_r = {}^{n-1}P_r + r.{}^{n-1}P_{r-1}.$$

The no. of permutations of n things taken r at a time $=$ (i) no. of those that do not contain a particular thing, say $a +$ (ii) no. of those that do contain it.

(i) The no. of permutations in this case is clearly equal to the no.

of permutations of $n-1$ things taken r at a time, i.e., $^{n-1}P_r$ since one thing is to be always excluded.

(ii) Leaving aside the particular thing a, the no. of permutations of the remaining $(n-1)$ things taken $r-1$ at a time, is $^{n-1}P_{r-1}$. Now since in any one of these arrangements, a can be placed in any one of the r positions relative to these $(r-1)$ things, we shall get from each such arrangement, r different permutations. Thus in this case the total no. of permutations $= r.^{n-1}P_{r-1}$.

Hence from (i) and (ii), we get the above result.

Note. This can also be easily established by applying the formula for ^nP_r.

Examples IX(A)

1. There are 6 colleges in a certain town ; in how many ways can a man send 3 of his sons to a college, so that no two of them may read in the same college ?

2. Suppose there are 5 steamers plying between Calcutta and Shibpur ; in how many ways can a man go from Calcutta to Shibpur and return by a different steamer ?

3. Five letters are written and five envelopes directed ; in how many ways can the letters be put in the envelopes ?

4. How many numbers each lying between 100 and 1000 can be formed with the digits 1, 3, 5, 7, 9, each of the digits occurring once and only once in each number ?

5. There are 16 stations on a local railway line. How many different kinds of single inter-class tickets must be printed in order that it may be possible to book from one station to another ?

6. (i) If $^{n-1}P_3 : {}^{n+1}P_3 = 5 : 12$; find n.

(ii) If $^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$; find n.

(iii) If $^{m+n}P_2 = 56$ and $^{m-n}P_2 = 12$; find m and n.

7. Show that

(i) $^nP_r = n.^{n-1}P_{r-1} = (n - r + 1).^nP_{r-1}.$

(i) $^1P_1 + 2.^2P_2 + 3.^3P_3 + \cdots + n.^nP_n = ^{n+1}P_{n+1} - 1.$

8. Show that 2.6.10.14... to n factors
$$= (n + 1)(n + 2)(n + 3)... \text{ to } n \text{ factors.}$$

9. In how many ways can the letters of the word *Monday* be arranged ? How many of these arrangements do not begin with M ? How many begin with M and do not end with y ?

10. (i) In how many ways can the letters of the word *Article* be arranged, so that the vowels may occupy only odd positions ?

(ii) Find how many different words can be formed with 5 given letters of which 3 are consonants and 2 are vowels, no two consonants being juxtaposed in any way.

(iii) Find how many words can be formed of the letters in the word *'failure'*, the four vowels always coming together. [*C. U. 1940*]

11. How many numbers between 4000 and 5000 can be formed with the digits 2, 3, 4, 5, 6, 7 ?

12. How many odd numbers of 5 significant digits can be formed with the digits 3, 2, 7, 4, 0 ?

13. Of the numbers formed by using all the figures 1, 2, 3, 4, 5, 6, 7 only once, how many are even ?

14. Find the total numbers of ways in which the letters of the word *mechanic* may be arranged, so that the vowels may never be separated.

15. In how many ways can 8 oranges of different sizes be distributed among 8 boys of different ages, so that the largest orange is always given to the youngest boy ?

16. (i) In how many ways can 6 examination papers be arranged, so that the best and the worst papers may never come together.

(ii) Show that the number of ways in which n boys can be arranged, so that the tallest and the shortest of them may never come together is $(n-2)\ \underline{|n-1}$.

17. In how many ways can 5 First year students and 3 Second year students be arranged, so that no two Second year students may sit together ?

18. If the number of 4-permutations of n things, in which one particular thing does not occur, is equal to the number in which it does occur, find n.

19. In how many of the permutations of 10 things taken 4 at a time, will one particular thing (i) always occur, (ii) never occur ?   [*C. U. 1936*]

20. In how many ways can 4 Arts students and 4 Science students be arranged alternately at a round table ?

21. In how many ways can 6 different beads be strung on a necklace ?

22. In how many ways can 7 persons sit at a round table, so that all shall not have the same neighbours in any two arrangements ?

23. In how many ways 5 men and 2 ladies be arranged at a round table if the two ladies (i) sit together, (ii) are separated ?

24. Find the number of arrangements that can be made out of the letters of the following words—

     (i) *India,*   (ii) *Calcutta,*

     (iii) *Examination,*   (iv) *Proportion,*

     (v) *Assassination,*   (vi) *Abracadabra.*

25. Show that the letters of the word *anticipation* can be arranged in three times as many ways as the letters of the word *commencement.*

26. A library has 5 copies of one book, 4 copies of each of two books, 6 copies of each of three books and single

copies of eight books. In how many ways can all the books be arranged ?

27. Show that the number of ways in which the letters of the word *arrange* can be arranged, so that two *r*'s do not come together is 900.

28. Show that the number of arrangements that can be made from the letters of the word *Orion*, so that the two consonants may not stand together is 36.

29. In how many ways can 3 prizes, one for good conduct, one for regular attendance and one for general proficiency, be given away to 10 boys ?

30. (i) How many numbers of not more than 4 digits can be formed with 3, 4, 5 ?

(ii) Find the sum of all the numbers that can be formed with the digits 1, 2, 3, 4 in the scale of 10.

31. In how many ways can a boat's crew of 8 men be arranged if 3 of the men can only row on stroke side and 2 others can only row on bow side.

32. Prove that the number of permutations of n different things taken r at a time, in which two specified things occur *next one another* is $2(n-1)(n-2)(n-3)\cdots(n-r+1)$.

ANSWERS

1. 120.   **2.** 20.   **3.** 120.   **4.** 60.   **5.** 240.

6. (i) 8.   (ii) 4.   (iii) $m=6, n=2$.   **9.** 720 ; 600 ; 96.

10. (i) 576.   (ii) 12.   (iii) 576.   **11.** 60.   **12.** 36.

13. 2160.   **14.** 2160.   **15.** 5040.   **16.** (i) 480.   **17.** 14400.

18. 8.   **19.** 2016 ; 3024.   **20.** 1152.   **21.** 60.   **22.** 2520.

23. (i) 240.   (ii) 480.   **24.** (i) 60.   (ii) 5040.   (iii) 4989600.

(iv) 151200.   (v) 10810800.   (vi) 83160.   **26.** $\lfloor 39 / \lfloor 5 \{\lfloor 4\}^2 \{\lfloor 6\}^3$.

29. 1000.   **30.** (i) 120.   (ii) 66660.   **31.** 1728.

SEC. B. COMBINATIONS

59. Combinations defined.

The different *groups* or *collections* that can be formed out of a given set of things by taking some or all of them at a time (without regard to the order of their arrangements) are called their *combinations*.

Thus, the combinations of the three letters a, b, c taken two at a time are ab, bc, ca.

The number of combinations of n different things taken r at a time is usually denoted by the symbol ^nC_r or $_nC_r$.

60. Combinations of things all different.

To find the number of combinations of n different things taken r at time $(r \leqslant n)$.

Let x denote the required number of combinations. If the r different things in each combination be arranged among themselves in all possible ways, each combination will produce $\lfloor r$ permutation.

$\therefore$ x combinations will produce $x \times \lfloor r$ permutations.

When all the things in each combination are arranged in all possible ways, all the possible permutations of n different things taken r at a time are obtained.

Hence, $x \lfloor r = {}^nP_r = n(n-1)(n-2)\cdots(n-r+1)$.

$\therefore$ x, i.e., $^nC_r = \dfrac{n(n-1)(n-2)\cdots\cdots(n-r+1)}{\lfloor r} \cdots$ (1)

Now, multiplying the numerator and the denominator of the fraction on the right side by $\lfloor n-r$, the above result can also be written in the following factorial from,

$$^nC_r = \frac{\lfloor n}{\lfloor r \rfloor n-r} \qquad \cdots \qquad \cdots \quad (2)$$

Cor. $^nC_1 = n$; $^nC_n = 1$; $^nP_r = {}^nC_r \times \lfloor r$.

Alternative method. (*Without assuming the formula for number of permutations.*)

Let the n different things be denoted by the n letters $a_1, a_2,\ldots\ldots a_n$. Each combination in which a particular letter say a_1 occurs, can be obtained by combining a_1 with each of the combinations of the remaining $(n-1)$ letters a_2, a_3, a_n taken $(r-1)$ at a time. Hence, the number of combinations in which a particular letter occurs is $^{n-1}C_{r-1}$.

Therefore, if all the combinations of the n letters taken r at a time be written down, every letter occurs in them $^{n-1}C_{r-1}$ times and hence the total number of letters in them is $n \times {}^{n-1}C_{r-1}$. But the total number of letters in them is also equal to $r \times {}^{n}C_r$, since each combination contains r letters and the total number of combinations is $^{n}C_r$.

$$\therefore \quad r \times {}^{n}C_r = n \times {}^{n-1}C_{r-1},$$

$$\text{or,} \quad {}^{n}C_r = \frac{n}{r} \times {}^{n-1}C_{r-1}.$$

Similarly, $\quad {}^{n-1}C_{r-1} = \dfrac{n-1}{r-1} \times {}^{n-2}C_{r-2}$

$$^{n-2}C_{r-2} = \frac{n-2}{r-2} \times {}^{n-3}C_{r-3}$$

$$\cdots \qquad \cdots \qquad \cdots$$

$$^{n-r+2}C_2 = \frac{n-r+2}{2} \times {}^{n-r+1}C_1$$

obviously, $\quad {}^{n-r+1}C_1 = \dfrac{n-r+1}{1}.$

Now, multiplying together the vertical columns and cancelling the common factors, we have

$$^{n}C_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{r(r-1)(r-2)\ldots 2.1}$$

$$= \frac{n(n-1)(n-2)\ldots(n-r+1)}{\lfloor r}.$$

61.   Complementary Combinations.

To prove that the number of combinations of n things taken r at a time, is equal to the number of combinations of n things taken $(n-r)$ at a time.

$$^nC_r = \frac{\lfloor n}{\lfloor r \ \lfloor n-r} \qquad \cdots \qquad \cdots \quad (1)$$

$$\therefore \quad ^nC_{n-r} = \frac{\lfloor n}{\lfloor (n-r) \ \lfloor \{n-(n-r)\}} = \frac{\lfloor n}{\lfloor n-r \ \lfloor r}.$$

$$\therefore \qquad ^nC_r = {}^nC_{n-r}.$$

Otherwise :

As often we select r things from n things, so often we leave $(n-r)$ things ; therefore the number of ways in which r things can be selected from n things is the same as the number of ways in which $(n-r)$ things can be selected from n things.   Hence the result.

Cor. If $^nC_r = {}^nC_{r'}$, then either $r = r'$ or $r + r' = n$, since $^nC_r = {}^nC_{n-r}$ and hence $r' = n - r$.

Note. In the above formula (1), if we put $r = n$, it becomes $^nC_n = \dfrac{\lfloor n}{\lfloor n \ \lfloor 0}$ and ^nC_n is obviously 1. Hence, in order that the formula may be intelligible in this case, the symbol $\lfloor 0$ *is considered as equivalent to* 1, though strictly speaking, it has no meaning.

Again in the formula $^nC_{n-r} = {}^nC_r$, if we put $r = n$, we have $^nC_o = {}^nC_n = 1$. Hence, the symbol ^nC_o *is considered as equivalent to* 1, though strictly speaking, it has no meaning.

62.   The relation $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$.

$$^nC_r + {}^nC_{r-1} = \frac{\lfloor n}{\lfloor r \ \lfloor n-r} + \frac{\lfloor n}{\lfloor r-1 \ \lfloor n-r+1}$$

$$= \frac{\lfloor n}{\lfloor r-1 \ \lfloor n-r} \left\{ \frac{1}{r} + \frac{1}{n-r+1} \right\}$$

$$= \frac{\lfloor n}{\lfloor r-1 \ \lfloor n-r} \cdot \frac{n+1}{r(n-r+1)}$$

$$= \frac{\lfloor n+1}{\lfloor r \ \lfloor n-r+1} = {}^{n+1}C_r.$$

Alternative proof:

The total number of combinations of $(n + 1)$ things taken r at a time may be divided into two groups (i) one including a certain particular thing and (ii) the other excluding that particular thing. Now the number of combinations in which the particular thing is included, is the number of combinations of the remaining n things taken $(r - 1)$ at a time, $i.e.$, $^{n}C_{r-1}$: and the number of combinations from which the particular things is excluded, is the number of combinations of the remaining n things taken r at a time, $i.e.$, $^{n}C_{r}$.

$$\therefore \quad ^{n}C_{r} + ^{n}C_{r-1} = {}^{n+1}C_{r}.$$

63. Restricted Combinations.

(i) *The number of combinations of n things taken r at a time in which p particular things always occur is* $^{n-p}C_{r-p}$.

(ii) *The number of combinations of n things taken r at a time in which p particular things never occur is* $^{n-p}C_{r}$.

(ii) If p particular things are first set aside, there will remain $(n - p)$ things, and hence if $(r - p)$ things be first selected in every possible way from these $(n - p)$ things and then added to the p particular things, we will get all the combinations in which p particular things always occur.

Hence, the required number $= {}^{n-p}C_{r-p}$.

(ii) Since p particular things are not to be included in any selection, if we set aside these p things from n things and then select r things out of the remaining $(n - p)$ things, we shall get all the combinations in which p particular things will never occur.

Hence, the required number $= {}^{n-p}C_{r}$.

64. Permutations of things not all different (*Alternative Proof*).

Let the n things be represented by n letters and suppose p of them are a's, q of them b's, r of them c's and the rest all different.

Then the number of ways in which n places can be filled up by these n letters, putting one in each place, is evidently the required number of permntations.

The p letters a can be put in the n places in ^nC_p ways, because there are n places to choose from. When the p letters a have oocupied their places in any one way, the q letters b can be put in the remaining $(n-p)$ places in $^{n-p}C_q$ ways. When the p letters a and q letters b have occupied their places in any one way, the r letters c can be put in the remaining $(n-p-q)$ places in $^{n-p-q}C_r$ ways. And the remaining $(n-p-q-r)$ different letters can be put in the remaining $(n-p-q-r)$ places in $\lfloor n-p-q-r$ ways.

[See *Art. 54, Cor. 1*]

Hence, the total number of ways in which the above n letters can fill up the n places

$$= {}^nC_p \times {}^{n-p}C_q \times {}^{n-p-q}C_r \times \lfloor n-p-q-r$$

$$= \frac{\lfloor n}{\lfloor p \lfloor n-p} \times \frac{\lfloor n-p}{\lfloor q \lfloor n-p-q} \times \frac{\lfloor n-p-q}{\lfloor r \lfloor n-p-q-r} \times \lfloor n-p-q-r$$

$$= \frac{\lfloor n}{\lfloor p \lfloor q \lfloor r}.$$

64. (a)   Illustrative Examples.

Ex. 1.   *Find the number of different triangles which can be formed by joining the angular points of a polygon of m sides. Show that this polygon has $\frac{1}{2}m\ (m-3)$ diagonals.*

A triangle can be formed by joining any three of the m points and these 3 points can be chosen in $^mC_3$ ways. Therefore the number of

triangles $= {}^mC_3 = \dfrac{m\ (m-1)(m-2)}{1.2.3} = \tfrac{1}{6}m\ (m-1)(m-2).$

The polygon has m angular points. The total number of lines that can be formed by joining any two of these points is $^mC_2$, *i.e.*, $\frac{1}{2}m\ (m-1)$.

But of these, the m sides of the polygon are not diagonals.

$\therefore$ The number of diagonals $= \frac{1}{2}m\ (m-1) - m = \frac{1}{2}m\ (m-3)$.

Ex. 2. *Prove that*

$$\frac{{}^{4n}C_{2n}}{{}^{2n}C_n} = \frac{1.3.5\cdots\cdots(4n-1)}{\{1.3.5\cdots\cdots(2n-1)\}^2}.$$

Here left side $= \dfrac{4n}{\lfloor 2n \lfloor 2n} \Big/ \dfrac{\lfloor 2n}{\lfloor n \lfloor n} = \dfrac{\lfloor 4n}{\lfloor 2n} \Big/ \left\{ \dfrac{\lfloor 2n}{\lfloor n} \right\}^2.$

We have $\lfloor 2n \,/\, \lfloor n = 2^n \{1.3.5\cdots\cdots(2n-1)\}.$ [See *Art. 58, Ex. 3*]

Now, writing $2n$ for n in the above result, we get

$$\lfloor 4n \,/\, \lfloor 2n = 2^{2n}(1.3.5\cdots\cdots\cdots\cdots(4n-1)\}.$$

$\therefore\ \dfrac{{}^{4n}C_{2n}}{{}^{2n}C_n} = \dfrac{2^{2n}\{1.3.5\cdots\cdots(4n-1)\}}{2^{2n}\{1.3.5\cdots\cdots(2n-1)\}^2} = \text{reqd. result.}$

Ex. 3. (i) *Find the number of different straight lines obtained by joining n different points on a plane, no three of which are collinear with the exception of p points which are collinear.*

(ii) *Find also the number of triangles formed by them.*

(i) If *no three points* were in the same straight line, the total number of straight lines would be ${}^nC_2 = \frac{1}{2}n\,(n-1)$. But p of the points being in the same straight line, $\frac{1}{2}p\,(p-1)$ straight lines are lost and in their place, we get only *one* straight line in which the p points are situated.

Hence, the number reqd. $= \frac{1}{2}n\,(n-1) - \frac{1}{2}p\,(p-1) + 1.$

(ii) Since a triangle is formed by joining any three points not in the same straight line, therefore reasoning as before, the required number of triangle $= {}^nC_3 - {}^pC_3 = \frac{1}{6}n\,(n-1)(n-2) - \frac{1}{6}p\,(p-1)(p-2).$

Ex. 4. *From* 10 *persons in how many ways can a selection of* 4 *be made* (i) *when one particular person is always* included *and* (ii) *when 2 particular persons are always* excluded.

(i) Since the particular person is to be *included* in every selection, we have only to choose 3 out of the remaining 9.

Hence, the no. of ways $= {}^9C_3 = \dfrac{9.8.7}{1.2.3} = 84.$

(ii) Since two particular persons are always to be *excluded*, we have to select 4 persons out of the remaining 8.

Hence, the no. of ways $= {}^8C_4 = \dfrac{8.7.6.5}{1.2.3.4} = 70.$

Ex. 5. *A candidate is required to answer 6 out of 10 questions which are divided into 2 groups each containing 5 questions, and he is not permitted to attempt more than 4 from any group. In how many different ways can he make up his choice ?* [*C. U. 1932*]

Let A and B be the two groups, each containing 5 questions. Now the candidate can select 4 from A and 2 from B or 3 from A and 3 from B or 2 from A and 4 from B. Hence the number of different ways in which he can make up his choice

$$= {}^5C_4 \times {}^5C_2 + {}^5C_3 \times {}^5C_3 + {}^5C_2 \times {}^5C_4$$

$$= {}^5C_1 \times {}^5C_2 + {}^5C_2 \times {}^5C_2 + {}^5C_2 \times {}^5C_1$$

$$= 5 \times \frac{5.4}{1.2} + \frac{5.4}{1.2} \times \frac{5.4}{1.2} + \frac{5.4}{1.2} \times 5$$

$$= 50 + 100 + 50 = 200.$$

Examples IX(B)

1. In an examination paper containing 10 questions, a candidate has to answer 7 questions only ; in how many ways can he choose the questions ?

2. A boy puts his hand into a bag which contains 10 differently-coloured marbles and brings out 3. How many different results are possible ?

3. In a boarding house, a different set of 5 boarders is appointed in the executive committee every week. If the number of boarders be 12, find how many weeks will elapse before the same set of 5 boarders will be in office again.

4. How many different triangles can be formed by joining the angular points of a decagon ?

Find also the number of the diagonals of the decagon ?

5. (i) If ${}^nP_r = 336$ and ${}^nC_r = 56$, find n and r.

(ii) If ${}^nC_{r-1} : {}^nC_r : {}^nC_{r+1} : : 2 : 3 : 4$, find n and r.

(iii) Show that the product of any r consecutive positive integers is exactly divisible by $\lfloor r.$

6. Prove that

(i) $^nC_r + {}^{n-1}C_{r-1} + {}^{n-1}C_{r-2} = {}^{n+1}C_r.$

(ii) $^nC_r + {}^{n-1}C_r + {}^{n-2}C_r + \cdots + {}^rC_r = {}^{n+1}C_{r+1}.$

7. Show that

$$\frac{^{2n}C_{2r}}{^nC_r} = \frac{(2n-1)(2n-3)(2n-5)\cdots\cdots(2n-2r+1)}{1.3.5\cdots\cdots\cdots(2r-1)}.$$

8. A person has got 12 acquaintances of whom 8 are relatives. In how many ways can he invite 7 guests so that 5 of them may be relatives ?

9. Out of 9 Swarajists and 6 Ministerialists, how many different committees can be formed, each consisting of 6 Swarajists and 3 Ministerialists ?

10. In how many ways can a committee of 3 ladies and 4 gentlemen be appointed from a meeting consisting of 18 ladies and 14 gentlemen ?

11. At an election there are 5 candidates and 3 members are to be elected and a voter is entitled to vote for any number to be elected. In how many ways may a voter choose a vote ?   [*C. U. 1935*]

12. From 6 gentlemen and 4 ladies, a committee of 5 is formed. In how many ways can this be done so as to include at least one lady ?   [*C. U. 1937*]

13. In a group of 15 boys there are 7 boy-scouts. In how many ways can 12 boys be selected so as to include (i) exactly 6 boy-scouts ; (ii) at least 6 boy-scouts ?
                                                   [*C. U. 1943*]

14. A cricket team consisting of 11 players is to be selected from 2 groups consisting of 6 and 8 players respectively. In how many ways can the selection be made on the supposition that the group of six shall contribute *no fewer than* 4 players ?   [*C. U. 1938*]

15. Out of 5 conservatives and 3 liberals, a committee of 6 is to be chosen. In how many ways can this be done

(i) when there are 4 conservatives in the committee;
(ii) when there is a majority of conservatives?

16. A certain council consists of a chairman, two vice-chairmen and twelve other members. How many different committees of 6 can be formed including always the chairman and only one vice-chairman?

17. Find the number of ways in which p positive signs and n negative signs may be placed in a row so that no two negative signs shall be together.

18. Out of 17 consonants and 5 vowels, how many different words can be formed, each consisting of 3 consonants and 2 vowels? [C. U. 1939]

19. Show that in $^{2n}C_n$, the number of combinations in which a particular thing occurs, is equal to the number in which it does not occur.

20. A committee of 6 is chosen from 10 Arts students and 7 Science students so as to contain *at least* 3 Arts students and 2 Science students. In how many different ways can this be done if two particular Science students refuse to serve on the same committee?

21. A football club is to play 10 football matches with the local teams. The results of these matches (win, loss or draw) are to be predicted. How many different forecasts can contain exactly 6 correct results?

ANSWERS

1. 120.   **2.** 120.   **3.** 792.   **4.** 120 ; 35.   **5.** (i) $n = 8$, $r = 3$.
    (ii) $n = 34$, $r = 14$.   **8.** 336.   **9.** 1680.   **10.** 816816.
11. 25.   **12.** 246.   **13.** (i) 196.   (ii) 252.   **14.** 344.
15. (i) 15.   (ii) 18.   **16.** 990.   **17.** $\lfloor p+1 / \lfloor n \lfloor p - n + 1$.
18. 816000. **20.** 7800.   **21.** 3360.

65.   Greatest Value of ^nC_r.

To find for what value of r the number of combinations of n things taken r at a time is greatest.

We have,

$$^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+2)(n-r+1)}{1.2.3\ldots(r-1)\,r}$$

$$^nC_{r-1} = \frac{n(n-1)(n-2)\cdots(n-r+2)}{1.2.3\ldots(r-1)}.$$

$$\therefore \quad ^nC_r = \frac{n-r+1}{r} \times {}^nC_{r-1}.$$

$$\therefore \quad ^nC_r > = \text{or} < {}^nC_{r-1},$$

according as $\dfrac{n-r+1}{r} > = \text{or} < 1$

i.e.,   as $n-r+1 > = \text{or} < r$

i.e.,   as $n+1 > = \text{or} < 2r$

i.e.,   as $r < = \text{or} > \tfrac{1}{2}(n+1)$.

(1) Let n be *odd* and $= 2m+1$; then

$$\tfrac{1}{2}(n+1) = m+1,$$

$$\therefore \qquad ^nC_r > = \text{or} < {}^nC_{r-1}$$

according as $r < = \text{or} > m+1$.

Thus, in the series $^nC_1, {}^nC_2, \ldots {}^nC_n$, so long as $r < m+1$, each term is greater than its preceding and when $r = m+1$ $^nC_r = {}^nC_{r-1}$ *i.e.*, $^nC_{m+1} = {}^nC_m$.   And for $r > m+1$, each term is less than its preceding.   Hence, ^nC_m and $^nC_{m+1}$ which are equal, are greater than any of the rest.   Hence ^nC_r is greatest when $r = m$ and $m+1$ *i.e.*, when $r = \tfrac{1}{2}(n-1)$ and $\tfrac{1}{2}(n+1)$.

(2) Let n be *even* and $= 2m$ say ; then $\frac{1}{2}(n+1) = m + \frac{1}{2}$.

$$\therefore \quad {}^nC_r > = \text{ or } < {}^nC_{r-1}$$

according as $r < = \text{ or } > m + \frac{1}{2}$.

Thus, in the series ${}^nC_1, {}^nC_2, \dots {}^nC_n$, so long as $r < m + \frac{1}{2}$, that is, for values of r from 1 up to m inclusive, each term is greater than its preceding and for values of r from $m + 1$ onwards, r being $> m + \frac{1}{2}$, each term is less than its preceding. Hence, ${}^nC_m$ is the greatest term.

Thus, ${}^nC_r$ is greatest when $r = m = \frac{1}{2}n$.

66. Total number of combinations of things all different.

To prove that the total number of combinations of n different things taken any number at a time is $2^n - 1$.

In making a selection out of the n things, each thing may be dealt with in 2 ways, for it may either be selected or left out. Since either way of dealing with any one thing may be combined with either way of dealing with each of the other $(n - 1)$ things, the total number of ways of dealing with n things is $2 \times 2 \times 2 \dots$ to n factors $= 2^n$.

But this includes the case in which all things are left out, which is inadmissible and hence rejecting this case, the required number of combinations $= 2^n - 1$.

Cor. We thus see that

$$ {}^nC_1 + {}^nC_2 + {}^nC_3 + \dots + {}^nC_n = 2^n - 1. $$

67. Total number of combinations of things not all different.

To prove that the total number of combinations of $(p + q + r + \cdots)$ things of which p are alike of one kind, q are alike of second kind, r are alike of third kind, and so on, taken any number at a time, is $(p+1)(q+1)(r+1)\cdots - 1$.

The p like things may be disposed of in $(p+1)$ ways ; for we may take either 1, 2, 3,... or p of them at a time or may not take any of them. Similarly q like things can be disposed of in $(q+1)$ ways; and so on. Since each of the $(p+1)$ ways of disposing of p like things may be combined with each of the $(q+1)$ ways of disposing of q like things, p and q like things may be disposed of in $(p+1)(q+1)$ ways. Similarly p, q, r like things may be disposed of in $(p+1)(q+1)(r+1)$ ways ; and so on. But this includes the case in which all the things are left out. Hence, rejecting this case, the total number of combinations

$$= (p+1)(q+1)(r+1)\cdots - 1.$$

68. Division into Groups.

(i) *To find the number of ways in which $(m+n)$ things can be divided into two groups containing m and n things respectively.*

The number of ways in which m things can be selected out of $(m+n)$ things $= {}^{m+n}C_m$ and each time that m things are taken, n things are left out to form the other group containing n, and this can be done in ${}^{n}C_n$ i.e., in one way only.

Hence, the required number of ways

$$= {}^{m+n}C_m = \frac{\lfloor m+n}{\lfloor m \lfloor m+n-m} = \frac{\lfloor m+n}{\lfloor m \lfloor n}.$$

Note. If $n=m$, the groups are equal and in this case the number of different ways of subdivision $= \dfrac{\lfloor 2m}{\{\lfloor m\}^2 \lfloor 2}$, since two groups can be interchanged without getting a new subdivision. If $2m$ things are to be distributed equally among two persons, the number of ways will however be $\dfrac{\lfloor 2m}{\{\lfloor m\}^2}.$

(ii) *To find the number of ways in which* $(m + n + p)$ *things can be divided into three groups containing* m, n *and* p *things respectively.*

The m things can be selected out of $(m + n + p)$ things in $^{m+n+p}C_m$ ways ; then the n things out of the remaining $(n + p)$ things in $^{n+p}C_n$ ways, and lastly the p things out of the remaining p things in ^pC_p *i.e.*, one way. Since each way of grouping m things can be associated with each way of grouping n things, we get the required number of ways

$$= {}^{m+n+p}C_m \times {}^{n+p}C_n$$

$$= \frac{\lfloor m+n+p}{\lfloor m \lfloor n+p} \times \frac{\lfloor n+p}{\lfloor n \lfloor p} = \frac{\lfloor m+n+p}{\lfloor m \lfloor n \lfloor p}.$$

Note 1. If $p = n = m$, the groups are equal and in this case the different ways of subdivision $= \dfrac{\lfloor 3m}{\{\lfloor m\}^3 \lfloor 3}$, for the three groups of subdivision in any way can be arranged in $\lfloor 3$ ways. If the $3m$ things are to be distributed equally among three persons, the number of wasys will be $\dfrac{\lfloor 3m}{\{\lfloor m\}^3}$.

Note 2. The above results and proof may easily be generalised for any number of groups.

Illustrative Examples

Ex. 1. *In how many ways is it possible to draw a sum of money from a bag containing a rupee, a four-anna piece, a two-anna piece, and an one-anna piece ?*

One, two, three or all the coins may be taken out.

∴ The reqd. no. of ways $= {}^4C_1 + {}^4C_2 + {}^4C_3 + {}^4C_4$

$$= 2^4 - 1 \text{ [See } Art.\ 66,\ Cor.\ ]$$

$$= 15.$$

Ex. 2. *In how many ways is it possible to make a selection by taking some or all 12 fruits, namely 5 cocoanuts, 3 oranges and 5 pine-apples ?*

Now out of $5+3+4$ *i.e.*, 12 fruits, 5 are alike of one kind, 3 are alike of second kind and 4 are alike of third kind.

∴ The reqd. no. of ways $= (5+1)(3+1)(4+1) - 1$

$$= 6.4.5 - 1 = 119.$$

Note. In the above method of proof, fruits of the same kind are all taken to be of the same shape. If however they are of different shapes we should apply Art. 66 instead of Art. 67 and in this case the reqd. no. of selections would be

$$(2^5 - 1) \times (2^3 - 1) \times (2^4 - 1) \ i.e., \ 31 \times 7 \times 15 = 3255,$$

provided we include at least one fruit of each kind.

Ex. 3. *Find the number of selections and arrangements that can be made by taking 4 letters from the word* expression.

Here we have 10 letters in all *viz.* $(e, e), (s, s), x, p, r, i, o, n$. Thus, we have in all 8 *different kinds of letters.* For a group of 4, the letters may be *classified* as follows :—

(i) Two alike, two others alike.

(ii) Two alike, two others different.

(iii) All four different.

From (i), we get 1 selection

(ii) we get $^2C_1 \times {}^7C_2$ *i.e.*, $2 \times 21 = 42$ selections.

(iii), we get $^8C_4$ *i.e.*, 70 selections.

∴ Total no. of *selections* $= 1 + 42 + 70 = 113$.

In finding the different arrangements of 4 letters, we have to permute in all possible ways each of the above three groups.

∴ Total no. of *arrangements*

$$= \frac{\lfloor 4}{\lfloor 2 \lfloor 2} + 42 \times \frac{\lfloor 4}{\lfloor 2} + 70 \times \lfloor 4$$

$$= 6 + 504 + 1680 = 2190.$$

Examples IX(C)

1. Prove that the greatest value of $^{2n}C_r$ is double the greatest value of $^{2n-1}C_r$.

2. A scout-master wishes to make up as many different parties as he can out of 10 boy-scouts, each party consisting of the same number. How many boys would there be in each party and how many different parties would be formed ?

3. The total number of combinations of $2n$ different things : the total number of combinations of n different things $= 65 : 1 :$ find the value of n.

4. From 4 mangoes, 3 oranges and 2 pine-apples how many selections of fruits can be made taking at least one of each kind.

[Here fruits of the same kind are of different shapes]

5. In how many ways can 5 oranges be divided between 2 boys, so that each may receive at least one orange ?

[Here oranges are of different shapes]

6. How many different factors can 210 have ?

7. In an examination a minimum is to be secured in each of 8 subjects for a pass. In how many ways can a student fail ?

8. Show that number of all possible selections of one or more questions from eight given questions, each question having an alternative is $3^8 - 1$.

9. If P_r denote the number of permutations of n different things, taken r at a time, show that

$$\frac{P_1}{1!} + \frac{P_2}{2!} + \frac{P_3}{3!} + \cdots + \frac{P_n}{n!} = 2^n - 1.$$

$$\left[Use \ \frac{^nP_r}{r!} = {^nC_r} \right]$$

10. Out of 3 rupees, 5 half-rupees and 6 four-anna pieces, in how many ways can a person subscribe to a famine relief fund ?

11. Prove that the total number of selections that can be made out of the letters '*daddy did a deadly deed*' is 1919.

12. The different sections of the Science Congress need the services of 3, 4, 5 volunteers respectively. If 12 students volunteer, in how many ways can they be allotted to different sections ?

13. Find the number of different ways of dividing mn things into n equal groups.

14. In how many ways can 12 apples be divided equally among 4 boys ?

15. Find the number of selections and arrangements that can be made by taking 4 letters from each of the words (i) *examination* and (ii) *parallelogram*.

16. If of $p + q + r$ things, p be alike, q be alike and the rest different, show that the total number of combinations is $(p + 1)(q + 1)2^r - 1$.

17. There are 12 balls of which 4 are green, 3 black and 5 white ; in how many ways can they be arranged so that no two white balls may occupy contiguous positions ?

18. If m parallel straight lines are intersected by n parallel lines, show that the number of parallelograms so formed is

$$\tfrac{1}{4}mn\,(m-1)(n-1).$$

ANSWERS

2. (i) 5. (ii) 252. **3.** 6. **4.** 315. **5.** 30. **6.** 15.

7. 255. **10.** 167. **12.** 27720. **13.** $\lfloor mn \div \{\lfloor m\}^n . \lfloor n$. **14.** 369600.

15. (i) 136, 2454. (ii) 150, 2510. **17.** 1960.

Examples IX(D)

[Additional Examples on Chapter IX]

1. The cylinder of a letter-lock contains 4 rings, each marked with 5 different letters. How many unsuccessful attempts to open the lock may be made by a person ignorant of the key-word ?

2. How many combinations can be formed of eight counters marked 1, 2, 3, 4, 5, 6, 7, 8 taking them 4 at a time, there being at least one odd and one even counter in each combination ? *[C. U. 1941]*

3. A committee of 7 is to be chosen from 13 students of which 6 are Arts students and 7 are Science students, in how many ways can the selection be made so as to always given a Science majority ?

4. A publisher proposes to issue a set of dictionaries to translate from any one language to any other. If he confines his plan to five languages, how many dictionaries must he publish ?

5. How many different algebraical quantities can be formed by combining a, b, c, d, e with the $+$ and $-$ signs, all the letters taken together ?

6. In how many ways can two sides of 6 players each be chosen from 12 men ?

7. In how many ways can a man wear 5 rings on the fingers (excepting the thumb) of one hand ?

8. In how many ways can the letters of the word "*pernicious*" be arranged without changing the order of the vowels ?

9. In how many ways can the letters of *pallmall* be arranged without letting all the l's come together ?

10. If n straight lines be drawn in a plane, no two being parallel and no three concurrent, how many points of intersection will there be ?

11. In how many ways can 52 cards be distributed among 4 players so that each may have 13 ?

12. There are 7 different situations vacant, of which 3 must be held by men and 2 by women ; the remaining two may be held by either men or women. If 6 male and 3 female candidates present themselves, in how many ways can the situations be filled up ?

13. Five men A, B, C, D, E are going to speak at a meeting ; in how many ways can they take their turns if (i) B does not speak before A, and (ii) A speaks immediately before B ?

14. How many numbers greater than a million can be formed with the digits 2, 5, 0, 5, 4, 2, 5 ?

15. Of the permutations of 10 things taken 7 at a time in how many do 3 particular things occur ?

16. Four letters are written and four envelopes are addressed. In how many ways can all the letters be placed in the wrong envelopes ?

17. Find the number of numbers less than 1000 and divisible by 5 which can be formed with the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, each digit occurring not more than once in each number.   [*C. U. 1942*]

18. In how many ways can three cards be selected from a pack of 52 cards, if at least one of them is to be an ace ?

19. A person wishes to make up as many different parties as he can out of 10 friends each party consisting of the same number. How many friends should he invite at a time and in how many of these parties would the same man be found ?

20. At a game of cards, 2 cards being dealt to each person, any one can have 1326 hands. Find the number of cards.

ANSWERS

1. 624.	**2.** 68.	**3.** 1058.	**4.** 20.	**5.** 32.	**6.** 462.
7. 6720.	**8.** 30239.	**9.** 780.	**10.** $\frac{1}{2}n\,(n-1)$.		

11. $\dfrac{\lfloor 52}{\{\lfloor 13\}^4}$.   **12.** 8640.   **13.** (i) 60. (ii) 24.   **14.** 360.   **15.** 176400.

16. 9.   **17.** 154.   **18.** 4804.   **19.** 5 ; 126.   **20.** 52.

10

BINOMIAL THEOREM

SEC. A. POSITIVE INTEGRAL INDEX

69. Introduction.

An expression containing two terms is called a binomial expression, thus, $x + y$, $x - a$ etc. are binomial expressions.

The Binomial Theorem[*] is a general algebraical formula by means of which any power of a binomial expression can be expressed as a series.

70. Binomial theorem for a Positive Integral Index.

To prove that when n *is a positive integer,*

$$(a+x)^n = a^n + {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 + \cdots \cdots$$
$$+ {}^nC_r a^{n-r}x^r + \cdots \cdots + x^n, \qquad \cdots \ (1)$$

$$= a^n + na^{n-1}x + \frac{n(n-1)}{\lfloor 2} a^{n-2}x^2 + \cdots \cdots$$
$$+ \frac{n(n-1)(n-2)\dots(n-r+1)}{\lfloor r} a^{n-r}x^r + \cdots \cdots + x^n. \ (2)$$

We have

$$(a + x)^n = (a + x)(a + x)\dots \text{ to } n \text{ factors.}$$

Every term in the above continued product of n factors is obtained by multiplying together n letters, one taken from each of the n factors and hence is of n dimensions. Thus, each term in the product being formed by multiplying together several a's and remaining x's and being of n dimensions, must be of the type $a^{n-r}x^r$ where r may have any value from 0 to n (the first term corresponding to $r = 0$ and the last term corresponding to $r = n$).

---

[*] It was first discovered by Newton.

Now each term involving $a^{n-r}x^r$ is obtained by taking x out of any r of the n factors and a out of the remaining $n-r$ factors. Therefore the number of terms which involve $a^{n-r}x^r$ must be equal to the number of ways in which r things can be selected out of n and hence the coefficient of $a^{n-r}x^r$ is ^nC_r and by giving to r the values $0, 1, 2,...n$ in succession, we obtain the coefficients of all the terms.

Hence $(a+x)^n = a^n + {^nC_1}a^{n-1}x + {^nC_2}a^{n-2}x^2 + \cdots\cdots$
$$+ {^nC_r}\,a^{n-r}\,x^r + \cdots + x^n \ [\ \because\ {^nC_0} = {^nC_n} = 1.]$$

Alternative Method *(Method of Induction)* :

By actual multiplication, we have

$$(a+x)^2 = a^2 + 2ax + x^2 = a^2 + {^2C_1}ax + x^2, \quad \cdots \quad \text{(i)}$$

$$(a+x)^3 = a^3 + 3a^2x + 3ax^2 + x^3$$
$$= a^3 + {^3C_1}a^2x + {^3C_2}ax^2 + x^3. \qquad \cdots \quad \text{(ii)}$$

The theorem is thus seen to be true when $n = 2$, and 3.

Let us *assume* therefore that the theorem is true when n has some particular value say m ; *i.e.*, let us suppose that

$$(a+x)^m = a^m + {^mC_1}a^{m-1}x + {^mC_2}a^{m-2}x^2 + \cdots$$
$$+ {^mC_r}a^{m-r}x^r + \cdots + x^m. \qquad \cdots \quad \text{(iii)}$$

Multiplying both sides of (iii) by $(a+x)$, we have

$$(a+x)^{m+1} = (a+x)\{a^m + {^mC_1}a^{m-1}x + \cdots$$
$$+ {^mC_r}a^{m-r}\,x^r + \cdots + x^m\}$$
$$= a^{m+1} + ({^mC_1} + 1)\,a^mx + ({^mC_2} + {^mC_1})\,a^{m-1}x^2$$
$$+ \cdots + ({^mC_r} + {^mC_{r-1}})\,a^{m-r+1}\,x^r + \cdots + x^{m+1}.$$

Since ${^mC_r} + {^mC_{r-1}} = {^{m+1}C_r}$ \hfill [See *Art. 62*]

and ${^mC_1} + 1 = m + 1 = {^{m+1}C_1}.$

$\therefore \quad (a+x)^{m+1} = a^{m+1} + {^{m+1}C_1}a^mx + {^{m+1}C_2}a^{m-1}x^2 + \cdots$
$$+ {^{m+1}C_r}\,a^{m-r+1}x^r + \cdots + x^{m+1}.$$

Thus, if the theorem is true for $n = m$, it is also true for $n = m + 1$. But it is true for $n = 3$; therefore it is true for $n = 4$, and being true for $n = 4$, it must be true for $n = 5$; and so on.

Hence, the theorem is true for all positive integral values of n.

Note 1. The *method of induction* consists in showing that, if the theorem is true for some particular integral value of n, say m, then it is true for $n = m + 1$. Consequently, if it is true for $n = 2$, then it is true for $n = 3$, and therefore for $n = 4$; and so on.

Note 2. For the sake of brevity, the notation $\binom{n}{r}$ is used to denote $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r\,!}$, where n is positive or negative, integer or fraction, r being a positive integer.

Obs. (i) When a quantity is expressed in a series, it is said to be *expanded*, and the right side of relation (1) or (2) above is called the *expansion* of $(a+x)^n$, and the coefficients, $^nC_0$, $^nC_1$, $^nC_2,\cdots{}^nC_r,\cdots{}^nC_n$, of the successive terms are called *Binomial coefficients*.

(2) It is clear from the expansion (1) that the *number of terms* in the expansion of $(a+x)^n$ is finite and is equal to $(n+1)$, *i.e.*, one more than the index of $(a+x)$.

(3) In every term the *index* of x is always *one less than the ordinal number of the term* and is the *same as the suffix of C* and the sum of the indices of a and x is n.

(4) From the expansion (2) it is clear that the number of factors both in the numerator and the denominator of each coefficient is one less than the ordinal number of the term.

70A. We give below an independent proof of the Binomial theorem in the form $(1+x)^n$ for the convenience of the students.

To prove that, when n *is a positive integer*,

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_r x^r + \cdots + x^n \qquad \cdots \ (1)$$

$$= 1 + nx + \frac{n(n-1)}{\underline{|2}} x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} x^r$$

$$+ \cdots + x^n \quad \cdots \ (2)$$

By actual multiplication, we have

$$(1+x)^2 = 1+2x+x^2 = 1+{}^2C_1x+x^2 \qquad \cdots \quad \text{(i)}$$

$$(1+x)^3 = 1+3x+3x^2+x^3 = 1+{}^3C_1x+{}^3C_2x^2+x^3 \qquad \cdots \quad \text{(ii)}$$

The theorem is thus easily seen to be true when $n=2$, and 3.

Let us *assume* therefore that the theorem is true when n has some particular value, say m, *i.e.*, let us suppose that

$$(1+x)^m = 1+{}^mC_1x+{}^mC_2x^2+\cdots+{}^mC_rx^r+\cdots+x^m. \qquad \text{(iii)}$$

Multiplying both sides of (iii) by $1+x$, we have

$$(1+x)^{m+1} = (1+x)\{1+{}^mC_1x+\cdots+{}^mC_rx^r+\cdots+x^m\}$$

$$= 1+({}^mC_1+1)x+({}^mC_2+{}^mC_1)x^2+\cdots$$

$$+({}^mC_r+{}^mC_{r-1})\,x^r+\cdots+x^{m+1}.$$

Since $\quad {}^mC_r+{}^mC_{r-1} = {}^{m+1}C_r \qquad\qquad$ [See *Art. 62*]

and $\quad {}^mC_1+1 = m+1 = {}^{m+1}C_1$.

$$\therefore \qquad (1+x)^{m+1} = 1+{}^{m+1}C_1x+{}^{m+1}C_2x^2+\cdots+{}^{m+1}C_rx^r+\cdots+x^{m+1}.$$

Thus, if the theorem is true for $n=m$, it is true for $n=m+1$. But it is true for $n=3$; therefore it is true for $n=4$ and being true for $n=4$, it must be true, $n=5$; and so on.

Hence the theorem is true for all positive integral values of n.

Note. It should be noted that the coefficient in the expansion of both $(a+x)^n$, and $(1+x)^n$ are same. In fact $(1+x)^n$ *is the particular case* of $(a+x)^n$, when $a=1$. Hence putting $a=1$ in the expansion of $(a+x)^n$, that of $(1+x)^n$ can be deduced.

Again, $\quad (a+x)^n = \left\{a\left(1+\dfrac{x}{a}\right)\right\}^n = a^n(1+y)^n$, where $y = \dfrac{x}{a}$

$$= a^n\{1+{}^nC_1y+{}^nC_2y^2+\cdots+y^n\}$$

$$= a^n\left\{1+{}^nC_1\,\dfrac{x}{a}+{}^nC_2\,\dfrac{x^2}{a^2}+\cdots+\dfrac{x^n}{a^n}\right\}$$

$$= a^n+{}^nC_1a^{n-1}x+{}^nC_2a^{n-2}x^2+\cdots+x^n.$$

Thus, the expansion of $(a+x)^n$ can be derived from that of $(1+x)^n$.

71. Particular Forms.

Since the expansions of $(a + x)^n$ and $(1 + x)^n$ are true for all values of x and a, hence writing $-x$ for x in the expansions of both, we have

$$(a - x)^n = a^n - {}^nC_1 a^{n-1}x + {}^nC_2 a^{n-2}x^2 - \cdots + (-1)^n x^n \cdots \text{(1)}$$
$$(1 - x)^n = 1 - {}^nC_1 x + {}^nC_2 x^2 - \cdots + (-1)^n x^n \qquad \cdots \text{(2)}$$

Thus, we see that in the expansions of $(a - x)^n$ and $(1 - x)^n$, the terms are *numerically* the same as those in the expansions of $(a + x)^n$ and $(1 + x)^n$ respectively, and are alternately positive and negative; and in both cases, the first term is positive, and the *last term* is *positive* or *negative* according as n is *even* or *odd*.

72. General Term.

The $(r + 1)$th term in the expansion of $(a + x)^n$ is usually called its *general term*, because any required term may be found from it, by giving a suitable value to r. The $(r + 1)$th term is sometimes shortly denoted by T_{r+1} or t_{r+1}.

In the expansion of $(a + x)^n$,

$$\text{2nd term} = {}^nC_1 a^{n-1}x$$
$$\text{3rd term} = {}^nC_2 a^{n-2}x^2$$
$$\text{4th term} = {}^nC_3 a^{n-3}x^3.$$
$$\therefore \quad (r + 1)\text{th term} = {}^nC_r a^{n-r} x^r.$$

Similarly, in the expansion of $(1 + x)^n$,

$$(r + 1)\text{th term} = {}^nC_r x^r.$$

Thus, *the general term in the expansion of* $(a + x)^n$

$$= {}^nC_r \; a^{n-r} \; x^r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} a^{n-r} \; x^r \quad \cdots \text{(1)}$$

Similarly, *the general term in the expansion of* $(1 + x)^n$

$$= {}^nC_r x^r = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} x^r. \qquad \cdots \text{(2)}$$

The general terms in the expansions of $(a - x)^n$ and $(1 - x)^n$ are respectively

$$(-1)^r \, {}^nC_r \, a^{n-r} \, x^r \text{ and } (-1)^r \, {}^nC_r \, x^r.$$

73. Middle Term.

To find the middle term (or terms) in the expansion of $(1 + x)^n$.

(i) Let n be *even* and $= 2m$, so that $m = \frac{1}{2}n$.

Since the total number of terms is $n + 1$, *i.e.*, $2m + 1$ (an odd number) *there will be only one middle term, viz.*, the $(m + 1)$th term, *i.e.*, the $(\frac{1}{2}n + 1)$th term. Hence,

the middle term is ${}^nC_{\frac{1}{2}n}.x^{\frac{1}{2}n}$, *i.e.*, $\dfrac{\underline{|n}}{\{|\frac{1}{2}n\}^2}.x^{\frac{1}{2}n}$

(ii) Let n be *odd* and $= (2m + 1)$, so that $m = \frac{1}{2}(n - 1)$.

Since the total number of terms is $n + 1$, *i.e.*, $2m + 2$ (an even number), *there will be two middle terms, i.e.*, the $(m + 1)$th and the $(m + 2)$th terms, *i.e.*, the $\{\frac{1}{2}(n - 1) + 1\}$th and $\{\frac{1}{2}(n + 1) + 1\}$th terms. Hence, *the two middle terms are*

$${}^nC_{\frac{1}{2}(n-1)} \, x^{\frac{1}{2}(n-1)} \text{ and } {}^nC_{\frac{1}{2}(n+1)} . \, x^{\frac{1}{2}(n+1)}, \, i.e.,$$

$$\dfrac{\underline{|n}}{|\frac{1}{2}(n-1). \, |\frac{1}{2}(n+1)} \, x^{\frac{1}{2}(n-1)} \text{ and } \dfrac{\underline{|n}}{|\frac{1}{2}(n+1). \, |\frac{1}{2}(n-1)} \, x^{\frac{1}{2}(n+1)}$$

The numerical coefficient of the two middle terms are the same.

74. Equidistant Terms.

In the expansion of $(a + x)^n$ *or* $(1 + x)^n$ *the coefficient of terms equidistant from the beginning and the end are equal.*

The coefficient of the $(r + 1)$th term from the beginning is ${}^nC_r$. Again, the $(r + 1)$th term from the end has

$\{(n + 1) - (r + 1)\}$, *i.e.*, $(n - r)$ terms before it and is therefore the $(n - r + 1)$th term from the beginning, and hence its coefficient is $^nC_{n-r}$.

But $^nC_r = {}^nC_{n-r}$.   [See *Art. 61*]

Hence the coefficients of the $(r + 1)$th term from the beginning = the coefficient of the $(r + 1)$th term from the end.

75.   Illustrative Examples.

Ex. 1. *Expand* $(1 - 2y)^6$.

Writing $-2y$ for x in the formula (1) of Art. 70A, we have
$(1 - 2y)^6 = 1 + {}^6C_1 (-2y) + {}^6C_2 (-2y)^2 + {}^6C_3 (-2y)^3 + \cdots$ to 7 terms.

Now, since $^nC_r = {}^nC_{n-r}$, we need calculate the coefficients only up to $^6C_3$. The rest may be written down at once; for $^6C_4 = {}^6C_2$, $^6C_5 = {}^6C_1$.

Since, $^6C_1 = 6, \ {}^6C_2 = \tfrac{1}{2}.6.5 = 15 \ ; \ {}^6C_3 = \tfrac{1}{6}.6.5.4 = 20.$

$\therefore \quad (1 - 2y)^6 = 1 + 6 (-2y) + 15 (-2y)^2 + 20 (-2y)^3 + 15 (-2y)^4$
$$+ 6 (-2y)^5 + (-2y)^6$$
$$= 1 - 12y + 60y^2 - 160y^3 + 240y^4 - 192y^5 + 64y^6.$$

Ex. 2. *Find the coefficients of x in the expansion of* $\left(x^2 + \dfrac{a^2}{x}\right)^5$.

Suppose x occurs in the $(r + 1)$th term ;

the $(r + 1)$th term $= {}^5C_r (x^2)^{5-r} \left(\dfrac{a^2}{x}\right)^r = {}^5C_r x^{10 - 3r}.a^{2r}.$

Since this term contains x. $\therefore x^{10 - 3r} = x.$

$\therefore \quad 10 - 3r = 1, \quad i.e., \ r = 3.$

$\therefore$ The 4th term contains x and we require its coefficient.

Now, the 4th term $= {}^5C_3 (x^2)^2. \left(\dfrac{a^2}{x}\right)^3 = {}^5C_3 x a^6 = 10 a^6 x.$

$\therefore$ The reqd. coefficient $= 10 a^6.$

Ex. 3. *Obtain the term free from x in the expansion of*

$$\left(x+\frac{1}{x}\right)^{2n}.$$ [*C. U. 1931*]

Let the $(r+1)$th term be free from x.

Now the $(r+1)$th term $= {}^{2n}C_r\, x^{2n-r}.\frac{1}{x^r} = {}^{2n}C_r\, x^{2n-2r}$.

In order that this term may be free from x, the index of x, namely $2n-2r$ must be zero, *i.e.*, $2n-2r=0$, $\therefore$ $r=n$.

Hence, the $(n+1)$th term is free from x ;

the $(n+1)$th term $= {}^{2n}C_n = \dfrac{\lfloor 2n}{\lfloor n \ \lfloor n}$, which is, therefore, the reqd. term.

Ex. 4. *Find the middle term of* $(a-2b)^8$.

Since there are 9 terms in the above expansion, the middle term is the 5th term.

$\therefore$ The reqd. middle term $= {}^8C_4\, a^4\,(-2b)^4$
$$= \frac{8.7.6.5}{1.2.3.4}\cdot a^4 2^4 b^4 = 1120 a^4 b^4.$$

Ex. 5. *Find the first five terms of the expansion of* $(1+x-2x^2)^7$ *in ascending powers of x.*

Putting $y = x - 2x^2$,
$(1+x-2x^2)^7 = (1+y)^7$

$$= 1 + 7y + \frac{7.6}{1.2}y^2 + \frac{7.6.5}{1.2.3}y^3 + \frac{7.6.5.4}{1.2.3.4}y^4 + \cdots$$

$$= 1 + 7\,(x-2x^2) + 21\,(x-2x^2)^2 + 35\,(x-2x^2)^3 + 35\,(x-2x^2)^4$$
$$+ \text{terms in higher powers of } x \text{ than } x^4$$

$$= 1 + 7\,(x-2x^2) + 21\,(x^2-4x^3+4x^4) + 35\,(x^3-6x^4+\cdots)$$
$$+ 35\,(x^4+\cdots) + \cdots$$

$$= 1 + 7x + 7x^2 - 49x^3 - 91x^4 \cdots\cdots$$

Ex. 6. *Prove that*

$$(1+x)^{2n} - 2nx\,(1+x)^{2n-1} + \frac{2n\,(2n-2)}{2\,!}\,x^2(1+x)^{2n-2}$$
$$- \frac{2n\,(2n-2)\,(2n-4)}{3\,!}\,x^3\,(1+x)^{2n-3} + \cdots \text{ to } (n+1) \text{ terms}$$
$$= (1-x^2)^n.$$

$$\text{Left side} = (1+x)^{2n} \left\{ 1 - n \cdot \left(\frac{2x}{1+x} \right) + \frac{n(n-1)}{2!} \cdot \left(\frac{2x}{1+x} \right)^2 - \cdots \right\}$$

$$= (1+x)^{2n} \left\{ 1 - \frac{2x}{1+x} \right\}^n = (1+x)^{2n} \left(\frac{1-x}{1+x} \right)^n = (1-x^2)^n.$$

Examples X(A)

1. Expand the following binomials :—

    (i) $(2a + 3b)^5$.
            (ii) $(a^2 - a \sqrt{b})^6$.

    (iii) $(x^2 - 1)^8$.
            (iv) $\left(x + \dfrac{1}{x} \right)^7$.

    (v) $\left(1 - \dfrac{1}{x} \right)^{10}$.
            (vi) $(1+x)^5 + (1-x)^5$.

2. Find the coefficient of x^8 in $(1+x^2)^{10}$.

3. Expand $\left(\dfrac{a}{b} + \dfrac{b}{a} \right)^{2n+1}$, giving in particular the general term and the two middle terms.   [*C. U. 1932*]

4. Find the term independent of x in the expansion of $\left(x^2 + \dfrac{1}{x} \right)^{12}$.   [*C. U. 1934*]

5. In the expansion of $(1+x)^{m+n}$ where m and n are positive integers. prove that the coefficients of x^m and x^n are equal.   [*C. U. 1935*]

6. Find the term independent of x in the expansion of $\left(2x + \dfrac{1}{3x^2} \right)^9$.   [*C. U. 1936*]

7. Find the value of

    (i) $(\sqrt{2}+1)^5 - (\sqrt{2}-1)^5$.

    (ii) $\{2 + \sqrt{(1-a)}\}^6 + \{2 - \sqrt{(2-a)}\}^6$.

8. (i) Find the two middle terms in the expansion of

    (a) $\left(x - \dfrac{1}{x}\right)^{13}$.   (b) $\left(x + \dfrac{1}{x}\right)^{2n+1}$.

  (ii) Find the middle terms in the expansion of

    (a) $\left(x - \dfrac{1}{x}\right)^{12}$.   (b) $\left(\dfrac{x}{b} + \dfrac{b}{x}\right)^{10}$.

    (c) $(1 - 2x + x^2)^n$.

9. Find the term independent of x in the expansion of

$$\left(1 + x\right)^p\left(1 + \dfrac{1}{x}\right)^q.$$

10. Find the coefficient of x in

$$(1 - 2x^3 + 3x^5)\left(1 + \dfrac{1}{x}\right)^{10}.$$

11. If in the expansion of $(1 + x)^{2n+1}$, the coefficient of x^r and x^{r+1} be equal, find r.   [*C. U. 1930*]

12. Find the term independent of x in $\left(1 + x\right)^2\left(x - \dfrac{1}{x}\right)^7$.

13. (i) Show that the coefficient of the $(p + 1)$th term in the expansion of $(1 + x)^{n+1}$ is equal to the sum of the coefficient of the pth and $(p + 1)$th terms in that of $(1 + x)^n$.

  (ii) Show that the coefficient of the middle term of $(1 + x)^{2n}$ is equal to the sum of the coefficients of the two middle terms of $(1 + x)^{2n-1}$.

14. Show that the middle term in the expansion of

  (i) $(1 + x)^{2n}$ is $\dfrac{1 . 3 . 5 \cdots (2n - 1)}{n!} \cdot 2^n x^n$.

  (ii) $\left(x - \dfrac{1}{x}\right)^{2n}$ is $\dfrac{1 . 3 . 5 \cdots (2n - 1)}{n!} \cdot (-2)^n$.   [*C. U. 1943*]

15. Find the first four terms of the expansion in ascending powers of x of $(1 + x + x^2)^6$.

16. Expand $(1 + x + x^2)^n$ in ascending powers of x as far as x^3.

17. Show that the coefficient of x^r in the expansion of $\left(x + \dfrac{1}{x}\right)^n$ is $\dfrac{\lfloor n}{\frac{1}{2}(n-r) \cdot \lfloor \frac{1}{2}(n+r)}$.

18. Find the coefficient of x^r in the expansion of
$$(x+2)^n + (x+2)^{n-1}(x+1) + (x+2)^{n-2}(x+1)^2 + \cdots$$
$$+ (x+1)^n.$$

19. Simplify
$$x^n(x-1)^n + {}^nC_1 x^{n-1}(x-1)^{n-1}(x+1) + \cdots\cdots$$
$$+ {}^nC_r x^{n-r}(x-1)^{n-r}(x+1)^r + \cdots\cdots + (x+1)^n.$$

20. Show that, if m is any positive integer,
$$m - \frac{m(m-1)}{1!} + \frac{m(m-1)(m-2)}{2!} - \cdots + (-1)^{m-1} m = 0.$$

21. If n is a positive integer, prove that
$$\left\{ 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots \right\}$$
$$\times \left\{ 1 + 2n + \frac{2n(2n-1)}{2!} + \frac{2n(2n-1)(2n-2)}{2!} + \cdots \right\}$$
$$= 1 + 3n + \frac{3n(3n-1)}{2!} + \frac{3n(3n-1)(3n-2)}{3!} + \cdots$$

22. Show that if n is a positive integer,
$$(1-x)^n = (1+x)^n - 2nx(1+x)^{n-1}$$
$$+ \frac{2n(2n-2)}{\lfloor 2} x^2 (1+x)^{n-2} - \cdots\cdots$$

23. Show that

$$\frac{x^r}{r\,!} + \frac{x^{r-1}a}{(r-1)\,!\,\,1\,!} + \frac{x^{r-2}a^2}{(r-2)\,!\,\,2\,!} + \cdots\cdots$$
$$+ \frac{xa^{r-1}}{1\,!\,(r-1)\,!} + \frac{a^r}{r\,!} = \frac{(x+a)^r}{r\,!}.$$

ANSWERS

1. (i) $32a^5 + 240a^4b + 720a^3b^2 + 1080a^2b^3 + 810ab^4 + 243b^5$.

   (ii) $a^{12} - 6a^{11}b^{\frac{1}{2}} + 15a^{10}b - 20a^9b^{\frac{3}{2}} + 15a^8b^2 - 6a^7b^{\frac{5}{2}} + a^6b^3$.

   (iii) $x^{16} - 8x^{14} + 28x^{12} - 55x^{10} + 70x^8 - 55x^6 + 28x^4 - 8x^2 + 1$.

   (iv) $x^7 + 7x^5 + 21x^3 + 35x + 35x^{-1} + 21x^{-3} + 7x^{-5} + x^{-7}$.

   (v) $1 - \frac{10}{x} + \frac{45}{x^2} - \frac{120}{x^3} + \frac{210}{x^4} - \frac{252}{x^5} + \frac{210}{x^6} - \frac{120}{x^7} + \frac{45}{x^8} - \frac{10}{x^9} + \frac{1}{x^{10}}$.

   (vi) $2(1 + 10x^2 + 5x^4)$.   2. 210.

3. $\left(\dfrac{a}{b}\right)^{2n+1} + {}^{2n+1}C_1\left(\dfrac{a}{b}\right)^{2n-1} + {}^{2n+1}C_2\left(\dfrac{a}{b}\right)^{2n-3} + \cdots + \left(\dfrac{b}{a}\right)^{2n+1}$.

   General term $= {}^{2n+1}C_r\left(\dfrac{a}{b}\right)^{2n-2r+1}$.

   Middle terms $= {}^{2n+1}C_n\,\dfrac{a}{b}$ and ${}^{2n+1}C_{n+1}\,\dfrac{b}{a}$.

4. 9th terms = 495.   6. 4th term = 1792/9.

7. (i) 82.   (ii) $2(365 - 363a + 63a^2 - a^3)$.

8. (i) (a) $1716x$, $-1716x^{-1}$.   (b) $\dfrac{\underline{|2n+1}}{\underline{|n}\,\underline{|n+1}}x$, $\dfrac{\underline{|2n+1}}{\underline{|n}\,\underline{|n+1}}\dfrac{1}{x}$.

   (ii) (a) 924.   (b) 252.   (c) $\dfrac{\underline{|2n}}{\underline{|n}\,\underline{|n}}\cdot(-1)^n x^n$.   9. $\dfrac{(p+q)\,!}{p\,!\,q\,!}$.

10. 540.   11. n.   12. 70.   15. $1 + 6x + 21x^2 + 50x^3$.

16. $1 + nx + \frac{1}{2}n(n+1)\,x^2 + \frac{1}{6}n(n-1)(n+4)\,x^3$.

18. $\dfrac{\underline{|n+1}}{\underline{|r}\,\underline{|n+1-r}}\{2^{n+1-r} - 1\}$.   19. $(x^2+1)^n$.

76. Greatest coefficient.

To find the greatest coefficient in the expansion of $(a+x)^n$ *or* $(1+x)^n$.

The coefficient of the $(r+1)$ th term in either of the expansion is ^nC_r. From Art. 65, we know that

(i) when n is even, ^nC_r is greatest, if $r = \frac{1}{2}n$;

and (ii) when n is odd, ^nC_r is greatest,

if $r = \frac{1}{2}(n-1)$, or, $\frac{1}{2}(n+1)$.

Hence when *n is even*, $(\frac{1}{2}n+1)$ th *i.e.*, the middle term has the greatest coefficient.

And when n is *odd*, $\{\frac{1}{2}(n-1)+1\}$ th, or, $\{\frac{1}{2}(n+1)+1\}$ th *i.e.*, the two middle terms have the greatest coefficients, the coefficients of these two terms being equal.

77. Greatest term.

To find the greatest term in the expansion of $(a+x)^n$, *where x is positive and a > 0.*

Let t_r denote the *rth* term in the expansion.

Then, $t_{r+1} = \dfrac{n(n-1)(n-2)\cdots(n-r+2)(n-r+1)}{1.2.3\cdots\cdots(r-1)\,r} \cdot a^{n-r}x^r$;

$t_r = \dfrac{n(n-1)(n-2)\cdots(n-r+2)}{1.2.3\cdots(r-1)} \cdot a^{n-r+1}x^{r-1}.$

$\therefore \quad \dfrac{t_{r+1}}{t_r} = \dfrac{n-r+1}{r} \cdot \dfrac{x}{a}.$

Thus $t_{r+1} > = $ or $< t_r$

according as $(n-r+1)x > = $ or $< ar$,

i.e., according as $(n+1)x > = $ or $< (x+a)r$

i.e.,   as $r < = $ or $> \dfrac{n+1}{x+a} \cdot x.$

Case I. Let $\dfrac{n+1}{x+a} \cdot x$ be an integer and $= p$ say.

Then, so long as $r < p$, $t_{r+1} > t_r$ *i.e.,* each succeeding term is greater than the preceding one so the terms go on increasing up to t_p.

When $r = p$, $t_{r+1} = t_r$, *i.e.,* $t_{p+1} = t_p$.

For values of $r > p$, (*i.e.,* from $p + 1$ onwards) $t_{r+1} < t_r$, and the succeeding terms therefore gradually diminish.

Hence $t_{p+1} = t_p$ and these are the greatest terms.

Case II. Let $\dfrac{n+1}{x+a} \cdot x$ be not an integer, and say $= q +$ a positive proper fraction, q being an integer.

Then, for values of r up to q, $r < \dfrac{n+1}{x+a} \cdot x$ and hence $t_{r+1} > t_r$.

And for values of $r = q + 1$ or greater, $t_{r+1} < t_r$.

Thus, $t_{q+1} > t_q > t_{q-1} \cdots$ and $t_{q+1} > t_{q+2} > t_{q+3} \cdots$

Hence, t_{q+1} is the greatest term.

Obs. If $(n+1)x/(x+a)$ is a proper fraction *i.e.,* $(n+1) x < (x+a)$ *i.e.,* $x < a/n$, the first term is the greatest and if $(n+1) x/(x+a) > n$ *i.e.,* $x > na$, the last term is the greatest.

Note 1. The greatest term in the expansion of $(1+x)^n$ can be deduced exactly in the same way ; only we have to write 1 for a.

Note 2. Since the numerically greatest term in the expansion of $(a-x)^n$ will be the same term as in the expansion of $(a+x)^n$, therefore in finding the *numerically greatest term in the expansion of* $(a-x)^n$, we should ignore the negative sign and proceed as above.

78.   Properties of Binomial coefficients.

For sake of simplicity, binomial coefficients $^nC_0$, $^nC_1$, $^nC_2$,... ^nC_r... etc. are often written as C_0, C_1, C_2,...C_r... Thus,

$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_{n-1}x^{n-1} + x^n \quad \cdots \ (1)$$

$$= C_0 + C_1 x + C_2 x^2 + \cdots + C_{n-1}x^{n-1} + C_n x^n \quad \cdots \ (2)$$

$$[\ \because\ {}^nC_0 = {}^nC_n = 1\ ]$$

(i) *In the expansion of* $(1+x)^n$, *the sum of the coefficients of all the terms is* 2^n.

Put $x = 1$ on both sides of (2).

$$\therefore \quad 2^n = C_0 + C_1 + C_2 + \cdots + C_n$$

$$= \text{sum of all the coefficients.}$$

(ii) *In the expansion of* $(1+x)^n$, *the sum of the coefficients of the odd term is equal to the sum of the coefficients of the even terms, each being equal to* 2^{n-1}.

Put $x = -1$ on both sides of (2).

$$\therefore \quad 0 = C_0 - C_1 + C_2 - \cdots + (-1)^n C_n.$$

$$\therefore\ C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots$$

$$= \tfrac{1}{2}\,.\,(\text{sum of all the coefficients})$$

$$= \tfrac{1}{2}\,.\,2^n = 2^{n-1}.$$

Cor.   From (i) above, it follows that

$$C_1 + C_2 + \cdots + C_n = 2^n - 1.$$

This show that the total number of combinations of n thing taken 1, 2, 3,...n at a time is $2^n - 1$.

Note. It should be noted that *the numerical coefficients in the expansion of* $(a+x)^n$ *has also the same two properties.* This can be easily shown by writing down the expansion of $(a+x)^n$ and putting $a = x = 1$ for the first case, and $a = 1$, $x = -1$ for the second case.

79. Illustrative Examples.

Ex. 1. *Find which is the numerically greatest term in the expansion of* $(5-4x)^{12}$, *when* $x=\frac{2}{3}$.

Here, $\qquad \dfrac{t_{r+1}}{t_r} = \dfrac{12-r+1}{r} \cdot \dfrac{4x}{5}$ *numerically*

$$= \dfrac{13-r}{r} \times \dfrac{8}{15}, \text{ since } x = \dfrac{2}{3}.$$

$\therefore \quad t_{r+1} > t_r$ if $8(13-r) > 15r$

$\qquad$ *i.e.,* if $104 > 23r$, *i.e.*, if $r < 4\frac{12}{23}$.

Hence, $t_{r+1} < t_r$, *i.e.*, $t_r > t_{r+1}$, if $r > 4\frac{12}{23}$.

$\therefore \quad t_5 > t_4 > t_3 \cdots$ and $t_5 > t_6 > t_7 \cdots$

$\therefore \quad$ 5th term is the greatest.

Ex. 2. *Find the value of the greatest term in the expansion of* $(1+2x)^6$, *when* $x=1$.

Here, $\dfrac{T_{r+1}}{T_r} = \dfrac{8-r+1}{r} \cdot 2x = \dfrac{9-r}{r} \cdot 2$, since $x=1$.

$\therefore \quad T_{r+1} > = \text{ or } < T_r$ according as $2(9-r) > = \text{ or } < r$

$\qquad\qquad$ *i.e.,* $\quad \cdots \qquad \cdots \quad 18-2r > = \text{ or } < r$

$\qquad\qquad$ *i.e.,* $\quad \cdots \qquad \cdots \qquad 18 > = \text{ or } < 3r$

$\qquad\qquad$ *i.e.,* $\quad \cdots \qquad \cdots \qquad\qquad r < = \text{ or } > 6.$

$\therefore \quad T_6 > T_5 > T_4 \cdots$ and $T_7 = T_6$ and $T_7 > T_8 > T_9 \cdots$

$\therefore \quad$ 6th and 7th terms which are equal, are the greatest.

$$T_6 = {}^8C_5 2^5 = {}^8C_3 2^5 = \dfrac{8.7.6}{1.2.3} \cdot 32 = 1792.$$

$\therefore \quad$ The value of the greatest terms is 1792.

Ex. 3. *Show that the greatest term in the expansion of* $(1+x)^{2m}$ *will have the greatest coefficient if x lies between* $\dfrac{m}{m+1}$ *and* $\dfrac{m+1}{m}$.

Since the greatest coefficient is that of the middle term, and since middle term is the $(m+1)$th term, there being altogether $(2m+1)$ terms,
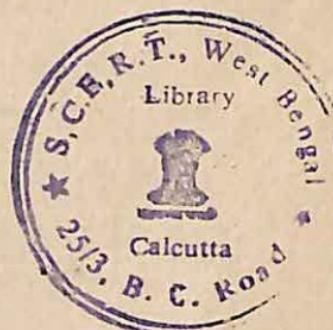
therefore we are to find the condition that the $(m+1)$th term will be the greatest term.

Since t_{m+1} is the greatest term. $\therefore$ $t_{m+1} > t_m$ and also $t_{m+1} > t_{m+2}$.

$$\because t_{m+1} > t_m ; \qquad \therefore {}^{2m}C_m\, x^m > {}^{2m}C_{m-1}\, x^{m-1} \quad \cdots \quad (1)$$

and $\quad \because t_{m+1} > t_{m+2} ; \quad \therefore {}^{2m}C_m\, x^m > {}^{2m}C_{m+1}\, x^{m+1} \quad \cdots \quad (2)$

$\therefore$ from (1), $\dfrac{2m-m+1}{m}\cdot x > 1.$ i.e., $\dfrac{m+1}{m}\cdot x > 1$ or $x > \dfrac{m}{m+1}$,

and from (2),

$$\frac{2m-(m+1)+1}{m+1}\cdot x < 1, \ i.e., \ \frac{m}{m+1}\cdot x < 1 \ \text{or} \ x < \frac{m+1}{m}.$$

Hence, x lies between $\dfrac{m}{m+1}$ and $\dfrac{m+1}{m}$.

Ex. 4. *If in the expansion of $(a+x)^n$, P be the sum of odd terms and Q the sum of even terms, show that*

(i) $P^2 - Q^2 = (a^2 - x^2)^n$.

(ii) $4PQ = (a+x)^{2n} - (a-x)^{2n}$.

Let $t_0, t_1, t_2, \ldots t_n$ represent the successive terms in the expansion of $(a+x)^n$.

Then, $\qquad (a+x)^n = t_0 + t_1 + t_2 + \cdots + t_n,$

and $\qquad (a-x)^n = t_0 - t_1 + t_2 - \cdots + (-1)^n\, t_n.$

$\therefore \qquad P = t_0 + t_2 + t_4 + \cdots$

$\qquad\qquad Q = t_1 + t_3 + t_5 + \cdots$

$\therefore \ P + Q = t_0 + t_1 + t_2 + t_3 + \cdots = (a+x)^n,$

$\qquad P - Q = t_0 - t_1 + t_2 - t_3 + \cdots = (a-x)^n,$

$\therefore$ Multiplying, $(P+Q)(P-Q) = (a-x)^n.(a-x)^n,$

$\qquad\qquad$ or $\quad P^2 - Q^2 = (a^2 - x^2)^n.$

Also, $\quad 4PQ = (P+Q)^2 - (P-Q)^2 = (a+x)^{2n} - (a-x)^{2n}.$

Ex. 5. *Prove that*

(i) $C_1 + 2C_2 + 3C_3 + \cdots + nC_n = n.2^{n-1}$.   [C. U. 1938]

(ii) $C_0{}^2 + C_1{}^2 + C_2{}^2 + \cdots + C_n{}^2 = \dfrac{\lfloor 2n}{\{\lfloor n\}^2}$.   [C. U. 1941]

(i) Left side

$$= n + 2 \cdot \frac{n(n-1)}{2} + 2 \cdot \frac{n(n-1)(n-2)}{2} + \cdots \text{ to } n \text{ terms}$$

$$= n + n(n-1) + \frac{n(n-1)(n-2)}{2} + \cdots\cdots + n$$

$$= n \left\{ 1 + (n-1) + \frac{(n-1)(n-2)}{2} + \cdots\cdots + 1 \right\}$$

$$= n(1+1)^{n-1} = n \cdot 2^{n-1}.$$

(ii) We have $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n.$ $\qquad \cdots$ (1)

Also $\qquad (x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} + \cdots + C_n.$ $\cdots$ (2)

Multiplying (1) by (2), we have

$(1+x)^{2n} =$ product of the two series on the right of (1) and (2).

Now, the coefficient of any power of x on the left = the coefficient of the same power of x on the right.

The coefficient of x^n in the product of the series (1) and (2),

$$= C_0{}^2 + C_1{}^2 + C_2{}^2 + \cdots + C_n{}^2.$$

Coefficient of x^n in the expansion of $(1+x)^{2n} = {}^{2n}C_n = \dfrac{\lfloor 2n}{\lfloor n \lfloor n}.$

Hence we get the required result.

Otherwise :

We have, $\quad (1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots\cdots + C_n x^n.$ $\qquad \cdots$ (1)

Changing x into $\dfrac{1}{x}$, we have

$$\left(1 + \frac{1}{x} \right)^n = C_0 + C_1 \cdot \frac{1}{x} + C_2 \cdot \frac{1}{x^2} + \cdots\cdots + C_n \cdot \frac{1}{x^n}. \quad \cdots \quad (2)$$

Multiplying (1) and (2), we have

$$(1+x)^n \cdot \left(1 + \frac{1}{x} \right)^n = \text{product of the series in (1) and (2)}$$

or, $\dfrac{(1+x)^{2n}}{x^n} =$ product of the series in (1) and (2). $\qquad \cdots$ (3)

Now, the term not involving x in the right side of (3),

$$= C_0{}^2 + C_1{}^2 + \cdots\cdots + C_n{}^2$$

and the term not involving x in the left side of (3),

$$= \text{coefficient of } x^n \text{ in } (1+x)^{2n} = {}^{2n}C_n = \frac{\lfloor 2n}{\lfloor n \ \lfloor n}$$

and since these two terms must be equal, the required result follows.

Ex. 6. *If m, n and r be positive integers, $r \not> m$ or n, prove that*
$${}^{m+n}C_r = {}^mC_r + {}^mC_{r-1}.{}^nC_1 + {}^mC_{r-2}.{}^nC_2 + \cdots + {}^mC_1.{}^nC_{r-1} + {}^nC_r.$$

We have

$$(1+x)^m = 1 + {}^mC_1 x + {}^mC_2 x^2 + \cdots\cdots + {}^mC_r x^r + \cdots\cdots + x^m \ ;$$
$$(1+x)^n = 1 + {}^nC_1 x + {}^nC_2 x^2 + \cdots\cdots + {}^nC_r x^r + \cdots\cdots + x^n.$$

Now, coefficient of x^r in the product of these two series

$$= {}^mC_r + {}^mC_{r-1}.{}^nC_1 + {}^mC_{r-2}.{}^nC_2 + \cdots\cdots + {}^nC_r$$
$$= \text{co-efficient of } x^r \text{ in the expansion of } (1+x)^m \times (1+x)^n$$
$$= \text{co-efficient of } x^r \text{ in the expansion of } (1+x)^{m+n}$$
$$= {}^{m+n}C_r.$$

Note. This result is known as *Vandermonde's* Theorem. It can also be easily established by the Theory of Combinations.

Examples X(B)

1. Find the greatest coefficient in the expansion of

(i) $(1+x)^8$.　　　(ii) $(1-x)^{11}$.　　　(iii) $(1+x)^{4n+3}$.

2. Find the greatest term in the expansion of

(i) $\left(1 + \dfrac{a}{x}\right)^{10}$, when $a = 1$, $x = 2$.

(ii) $(1 + \tfrac{5}{16}x)^{12}$, when $x = 2$.

(iii) $(ax - by)^{10}$, if $a = 2$, $b = 5$, $x = 3$, $y = \tfrac{1}{2}$.

3. Find the value of the greatest term in the following expansion :

(i) $(1+x)^6$, when $x = \tfrac{1}{2}$.

(ii) $(1 - \tfrac{2}{3}x)^8$, when $x = 1\tfrac{1}{2}$.

(iii) $(1 - 2x)^{13}$, when $x = \tfrac{1}{30}$.

4. (i) Show that the greatest term in the expansion of $(1+x)^{2n+1}$ has also the greatest coefficient if x lies between $\dfrac{n}{n+2}$ and $\dfrac{n+2}{n}$.

(ii) Find the limits within which x must lie in order that the greatest term in the expansion of $(1+x)^{12}$ may have the greatest coefficient.

5. The coefficients of the second, third and fourth terms of the expansion of $(1+x)^n$ are in Arithmetical Progression ; find n.

6. (i) If three successive coefficients in the expansion of $(1+x)^n$ be 220, 495 and 792 ; find n.

(ii) If a, b, c be the three consecutive coefficients of the expansion of a power of $(1+x)$ prove that the index of the power is $\dfrac{2ac+b(a+c)}{b^2-ac}$.

7. If the pth, $(p+1)$th and $(p+2)$th coefficients of $(1+x)^n$ are in A. P., shew that $n^2 - n(4p+1) + 4p^2 - 2 = 0$.

8. In the expansion of $(1+x)^{25}$, the coefficients of the $(2r+1)$th and $(r+5)$th terms are equal ; find r.

9. If two successive coefficients of an expanded binomial are equal, show that the next preceding and the next succeeding coefficients are also equal.

10. If a_1, a_2, a_3, a_4 are any four consecutive coefficients of an expanded binomial, show that

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{2a_2}{a_2+a_3}.$$

11. Prove that the difference between the coefficients of x^{r+1} and x^r in the expansion of $(1+x)^{n+1}$ is equal to the difference between the coefficients of x^{r+1} and x^{r-1} in the expansion of $(1+x)^n$.

12. If a, b, c, d be the 6th, 7th, 8th and 9th terms in the expansion of $(1+x)^n$, prove that $\dfrac{b^2-ac}{c^2-bd} = \dfrac{4a}{3c}$.

13. If P_n denote the product of all the coefficients in the expansion of $(1 + x)^n$, show that

$$\frac{P_{n+1}}{P_n} = \frac{(n+1)^n}{n!}.$$

14. If $(1 + x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$, prove that

(i) $C_1 - 2C_2 + 3C_3 - \cdots\cdots + (-1)^{n-1} nC_n = 0.$

(ii) $C_0 + 2C_1 + 3C_2 + \cdots\cdots + (n+1)C_n = (n+2).2^{n-1}.$

(iii) $C_0 + 3C_1 + 5C_2 + \cdots\cdots + (2n+1)C_n = (n+1)2^n.$

(iv) $\dfrac{C_0}{1} + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \dfrac{C_3}{4} + \cdots\cdots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1} - 1}{n+1}.$

$$[\ C.\ U.\ 1945\ ]$$

(v) $\dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \cdots\cdots + \dfrac{nC_n}{C_{n-1}} = \dfrac{n(n+1)}{2}.$

(vi) $(C_0 + C_1)(C_1 + C_2)\cdots\cdots (C_{n-1} + C_n)$
$$= \frac{(n+1)^n}{\lfloor n} \cdot C_1 C_2 \cdots\cdots C_n.$$

(vii) $C_0 C_n + C_1 C_{n-1} + \cdots\cdots + C_n C_0 = \dfrac{\lfloor 2n}{\lfloor n \lfloor n}.$

(viii) $C_0 C_r + C_1 C_{r+1} + C_2 C_{r+2} + \cdots\cdots + C_{n-r} C_n$
$$= \frac{\lfloor 2n}{\lfloor n-r \lfloor n+r}.$$

(ix) $C_1{}^2 + 2C_2{}^2 + 3C_3{}^2 + \cdots + nC_n{}^2 = \dfrac{\lfloor 2n-1}{\lfloor n-1 \lfloor n-1}.$

(x) $C_0{}^2 - C_1{}^2 + C_2{}^2 - \cdots\cdots + (-1)^n C_n{}^2 = 0,$

or $(-1)^{\frac{n}{2}} \dfrac{\lfloor n}{\{\lfloor \frac{1}{2} n\}^2}$ according as n is odd or even.

15. Show that $(C_0 + C_1 + C_2 + \cdots\cdots + C_n)^2$
$$= {}^{2n}C_0 + {}^{2n}C_1 + {}^{2n}C_2 + \cdots\cdots + {}^{2n}C_{2n}.$$

16. (i) If $(1 + x + x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots\cdots + a_{2n} x^{2n}$, show that

(a) $a_0 + a_1 + a_2 + \cdots\cdots + a_{2n} = 3^n$.

(b) $a_0 - a_1 + a_2 - \cdots\cdots + a_{2n} = 1$.

(ii) If $t_0, t_1, t_2, \ldots\ldots t_n$ represent the successive terms in the expansion of $(a + x)^n$, show that

$$(t_0 - t_2 + t_4 - \cdots)^2 + (t_1 - t_2 + t_3 - \cdots)^2 = (a^2 + x^2)^n.$$

17. Show that, if m is a positive integer

$$2^m - \frac{m}{1!} \cdot 2^{m-1} + \frac{m(m-1)}{2!} \cdot 2^{m-2} - \cdots + (-1)^m = 1.$$

18. Show that, if n is a positive integer,

$$1 + 2n + \frac{2n(2n-1)}{2!} + \frac{2n(2n-1)(2n-2)}{3!} + \cdots$$

$$= \left\{ 1 + n + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \cdots \right\}^2.$$

19. Prove that if m is a positive integer,

$$x - {}^mC_1 (x + y) + {}^mC_2 (x + 2y) - {}^mC_3 (x + 3y) + \cdots = 0.$$

20. Prove that if n is a positive integer,

$$1 - {}^nC_1 \frac{1+x}{1+nx} + {}^nC_2 \frac{1+2x}{(1+nx)^2} - {}^nC_3 \frac{1+3x}{(1+nx)^3} + \cdots = 0.$$

21. Apply the Binomial Theorem to find the value of

(i) $(98)^3$.

(ii) $(.999)^3$ correct to 3 places of decimals.

ANSWERS

1. (i) 70.   (ii) 462.   (iii) $\underline{4n+3} / \underline{2n+2} \, \underline{2n+1}$.

2. (i) 4th term.   (ii) 5th and 6th terms.   (iii) 4th term.

3. (i) $3\frac{3}{4}$.   (ii) 70.   (iii) 1.   4. (ii) $\frac{6}{7} < x < \frac{7}{6}$.   5. 7.

6. (i) 12.   8. 7.   21. (i) 941192.   (ii) .997.

SEC. B. FRACTIONAL OR NEGATIVE INDEX

80. Introduction.

It has been shown that when n is a *positive integer*, the Binomial Series

$$1 + nx + \frac{n(n-1)}{\underline{|2}} x^2 + \cdots + \frac{n(n-1)\cdots(n-r+1)}{\underline{|r}} x^r + \cdots$$

terminates after $(n+1)$ terms.

For if in the general term, values *greater* than n are given to r, one of the factors in the numerator of the coefficient

$$\frac{n(n-1)\cdots(n-r+2)(n-r+1)}{\underline{|r}} \text{ vanishes.}$$

∴ The coefficient of every power of x higher than the nth becomes zero. Hence the series terminates at x^n.

But if in the above series, n be *fractional* or *negative*, then since r is a positive integer ($r+1$ being the ordinal number of a term) r can never have such a value whereby any factor in the numerator of the coefficient of the general term vanishes. Therefore in such a case, however large the power of x is, its coefficient would never vanish and hence the above series would not terminate and would become an *infinite* series.

Now, infinite series cannot be properly and adequately treated without having recourse to the principle of convergency and divergency and a complete rigorous proof of the Binomial Theorem for fractional or negative index as well as of the Exponential Theorem is beyond the scope of an elementary treatise.

81. Binomial Theorem for fractional or negative index.

To prove that when n is negative or fractional,

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}} x^2 + \frac{n(n-1)(n-2)}{\underline{|3}} x^3 + \cdots\cdots$$

$$+ \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} x^r + \cdots\cdots \text{ to } \infty \quad \cdots \quad (1)$$

provided x *is numerically less than* 1.

Let $f(n)$ denote the series on the right side of (1),

then $f(m) = 1 + mx + \dfrac{m(m-1)}{\lfloor 2} x^2 + \cdots$ ... (2)

$f(n) = 1 + nx + \dfrac{n(n-1)}{\lfloor 2} x^2 + \cdots$ ... (3)

and $f(m+n) = 1 + (m+n)x + \dfrac{(m+n)(m+n-1)}{\lfloor 2} x^2 + \cdots$ (4)

Now, when m and n are positive integers, by the Binomial Theorem for a positive integral index, we have

$$f(m) = (1+x)^m \; ; \; f(n) = (1+x)^n$$

$$\text{and } f(m+n) = (1+x)^{m+n}.$$

But $(1+x)^m \times (1+x)^n = (1+x)^{m+n}$ evidently.

∴ When m and n are positive integers,

$$\left\{ 1 + mx + \frac{m(m-1)}{\lfloor 2} x^2 + \cdots \right\}\left\{ 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \cdots \right\}$$

$$= 1 + (m+n)x + \frac{(m+n)(m+n-1)}{\lfloor 2} x^2 + \cdots \qquad (5)$$

Now when any two algebraical expressions containing a *finite* number of terms are multiplied together, the *form* of the product is *always* the same, whatever values positive or negative, integral or fractional, the symbols involved in the expression may have, *provided the number of terms in the two expressions does not become infinite* for any such values of the symbols.[*] If the number of terms does become infinite, then the form of the product will remain the same *provided a certain condition is fulfilled.*[†] In relation (5),

[*] This is known as the "Principle of the Permanence of algebraic forms."

[†] Here the question of convergency arises, which is beyond the scope of the present treatise.

when fractional or negative values are given to m and n in the series on the left side, the number of terms in both the series become infinite, still the form of the product will remain the same, since x is numerically less than unity.

Hence, *if x is numerically less than 1,*

the relation (5) will be true for all values of m and n,

i.e., $f(m) \times f(n) = f(m + n)$, for all values of m and n,

$\therefore \quad f(m) \times f(n) \times f(p) = f(m + n) \times f(p) = f(m + n + p).$

Similarly, $f(m) \times f(n) \times f(p) \cdots$ to a finite number of factors

$$= f(m + n + p + \cdots \text{ to a finite number of terms}), \cdots \quad (6)$$

for all values of $m, n, p, \ldots$

(i) Let n be any positive rational fraction $= \dfrac{h}{k}$, where h and k are positive integers,

$\therefore$ By the relation (6), we have

$$f\left(\frac{h}{k}\right) \times f\left(\frac{h}{k}\right) \times f\left(\frac{h}{k}\right) \times \cdots \text{ to } k \text{ factors}$$

$$= f\left(\frac{h}{k} + \frac{h}{k} + \frac{h}{k} + \cdots \text{ to } k \text{ factors}\right);$$

i.e., $\left\{ f\left(\dfrac{h}{k}\right) \right\}^k = f(h)$

$$= 1 + hx + \frac{h(h-1)}{\lfloor 2} x^2 + \cdots\cdots$$

$$= (1 + x)^h, \text{ since } h \text{ is positive integer.}$$

$$\therefore \quad (1 + x)^{\frac{h}{k}} = f\left(\frac{h}{k}\right) = 1 + \frac{h}{k} x + \frac{\dfrac{h}{k}\left(\dfrac{h}{k} - 1\right)}{\lfloor 2} x^2 + \cdots\cdots$$

Thus, the theorem is proved when n is positive rational fraction, provided $-1 < x < 1$.

(ii) Let n be a negative rational number (integral or fractional) and equal to $-s$, where s is positive.

Then by (5), $f(-s) \times f(s) = f(-s+s) = f(0) = 1$.

$$\therefore \quad f(-s) = \frac{1}{f(s)} = \frac{1}{(1+x)^s}, \text{ since } s \text{ is positive}$$
$$= (1+x)^{-s}.$$

$$\therefore \quad (1+x)^{-s} = 1 - sx + \frac{(-s)(-s-1)}{\lfloor 2} x^2 + \cdots\cdots$$

This proves the Binomial Theorem for any negative index, provided $-1 < x < 1$.

Thus, the Binomial Theorem is completely established for fractional or negative index, when x is numerically less than unity.

Obs. When x is numerically equal to unity, the Binomial Theorem is true under certain conditions but the proper treament of these cases is beyond the scope of this elementary work. *When x is greater than unity, the Binomial Theorem is not true.* This is easily seen by putting $x = 2$ in the expansion of $(1-x)^{-1}$. Thus,

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

Putting $x = 2$, we get $-1 = 1 + 2 + 2^2 + 2^3 + \cdots$, which is absurd, since left side is negative and right side is positive.

Note 1. The statement that "x is numerically less than 1" is sometimes expressed by the notation $|x| < 1$.

Note 2. It should be carefully noted that the successive coefficients in the expansion of $(1+x)^n$, when n is not a positive integer, cannot be denoted by the symbols $^nC_0$, $^nC_1$, $^nC_3$,..., ^nC_r,..., for in these symbols n stands for a positive integer, whereas the n in $(1+x)^n$ is either a fraction or a negative number.

Note 3. It would not be out of place to mention here that the Binomial Theorem for fractional or negative index has sometimes been established by an improper use of what is known as the Principle of the Permanence of algebraic forms, thereby leading to the absurd result that the Binomial Theorem is true for all values of x even when

the index is fractional or negative. The fallacy lies in the fact that the Principle of the Permanence of forms which is, strictly speaking, applicable to those cases only where the number of terms in the expression does not become infinite for any values of the symbols involved, is extended without restriction, to the case where the number of terms does become infinite for some values of the symbols.

82.　Vandermonde's Theorem.

If r be a positive integer and m and n have any values whatever, then

$$(m+n)_r = m_r + {}^rC_1 m_{r-1} n_1 + {}^rC_2 m_{r-2} n_2 + \cdots\cdots + n_r$$

$$= m_r + r m_{r-1} n_1 + \frac{r(r-1)}{1.2} m_{r-2} n_2 + \cdots\cdots + n_r \quad \cdots \quad (1)$$

where the notation m_r stands for

$$m(m-1)(m-2)\ldots(m-r+1).$$

When $r=1$, $(m+n)_1 = m+n = (m_1 + n_1)$ obviously.

Multiply both sides by $m+n-1$,

$$\therefore \ (m-n)_1(m+n-1) = m(m+n-1) + n(m+n-1)$$
$$= m(m-1) + mn + mn + n(n-1)$$
$$= m_2 + 2m_1 n_1 + n_2,$$
$$i.e., \quad (m+n)_2 = m_2 + {}^2C_1 m_1 n_1 + n_2.$$

Thus the theorem is true when $r=2$.

Let us *assume* that the theorem is true when r has any particular value, say p. Then, we have

$$(m+n)_p = m_p + {}^pC_1 m_{p-1} n_1 + {}^pC_2 m_{p-2} n_2 + \cdots + n_p. \quad \cdots \quad (2)$$

Multiply both sides by $m+n-p$.

$$\therefore \ (m+n)_p(m+n-p)$$
$$= m_p(\overline{m-p}+n) + {}^pC_1 m_{p-1} n_1 (\overline{m-p+1}+\overline{n-1})$$
$$\quad + {}^pC_2 m_{p-2} n_2 (\overline{m-p+2}+\overline{n-2}) + \cdots + n_p(\overline{n-p}+m)$$
$$= (m_{p+1} + m_p n_1) + {}_pC_1 (m_p n_1 + m_{p-1} n_1)$$
$$\quad + {}^pC_2 (m_{p-1} n_2 + m_{p-2} n_3) + \cdots + n_{p+1}$$

$$= m_{p+1} + m_p n_1 \left(1 + {}^pC_1\right) + m_{p-1}n_2 \left({}^pC_1 + {}^pC_2\right)$$
$$+ \cdots + m_{p-s} \cdot n_{s+1} \left({}^pC_s + {}^pC_{s+1}\right) + \cdots + n_{p+1} \; ;$$

$(m+n)_{p+1}$

$$= m_{p+1} + {}^{p+1}C_1 m_p n_1 + {}^{p+1}C_2 m_{p-1}n_2 + \cdots + n_{p+1}$$
$$\left[\because \quad {}^pC_s + {}^pC_{s+1} = {}^{p+1}C_{s+1}. \right]$$

Thus, if the result is true for any value of r, it is true for the next higher value of r; but it is true for $r = 1, 2$; and hence it is true for $r = 3$, 4 etc. and it is thus true universally.

83. Alternative Proof of the Binomial theorem for fractional or negative index.

To prove that if n is negative or fractional

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{\underline{|2}} x^2 + \cdots + \frac{n(n-1)(n-2)\cdots(n-r+1)}{\underline{|r}} x + \cdots$$

provided x is numerically less than 1. $\qquad \cdots \quad (1)$

Let $f(n)$ denote the series on the right side of (1) and let n_r denote $n(n-1)(n-2)\cdots(n-r+1)$, then

$$f(n) = 1 + n_1 x + \frac{n^2}{\underline{|2}} x^2 + \cdots + \frac{n_r}{\underline{|r}} x^r + \cdots \qquad \cdots \quad (2)$$

$$\therefore \; f(m) = 1 + m_1 x + \frac{m_2}{\underline{|2}} x^2 + \cdots + \frac{m_r}{\underline{|r}} x^r + \cdots \quad \cdots \quad (3)$$

and $f(m+n) = 1 + (m+n)_1 x + \dfrac{(m+n)_2}{\underline{|2}} x^2 + \cdots$

$$+ \frac{(m+n)_r}{\underline{|r}} x^r + \cdots \qquad \cdots \qquad \cdots \quad (4)$$

Now, since x is numerically less than 1, multiplying together (2) and (3) and picking out the coefficients of the

successive powers of x, we see that[*] the coefficient of x^r in the product of $f(m)$ and $f(n)$

$$= \frac{m_r}{\lfloor r} + \frac{m_{r-1}}{\lfloor r-1} \cdot \frac{n_1}{\lfloor 1} + \frac{m_{r-2}}{\lfloor r-2} \cdot \frac{n_2}{\lfloor 2} + \cdots + \frac{n_r}{\lfloor r}$$

$$= \frac{1}{\lfloor r} \left\{ m_r + r m_{r-1} n_1 + \frac{r(r-1)}{\lfloor 2} m_{r-2} n_2 + \cdots + n_r \right\}$$

$$= \frac{1}{\lfloor r} (m+n)_r, \; [\; By \; Vandermonde's \; Theorem. \;]$$

$\therefore f(m) \times f(n)$

$$= 1 + (m+n)_1 x + \frac{(m+n)_2}{\lfloor 2} x^2 + \cdots + \frac{(m+n)_r}{\lfloor r} x_r + \cdots$$

$$= f(m+n).$$

Thus, *when x is numerically less than 1,*

$$f(m) \times f(n) = f(m+n)$$

for all values of m and n.

Now, the rest of the proof is the same as that given after the relation (5) in Art. 81.

84. General Term.

The $(r+1)$th term is usually called the general term. Let us denote it shortly by T_{r+1}.

Since $(1+x)^n = 1 + nx + \frac{n(n-1)}{\lfloor 2} x^2 + \frac{n(n-1)(n-2)}{\lfloor 3} x^3 + \cdots$

$$\therefore T_{r+1} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} x^r.$$

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[*] It is given in text-books on Higher Algebra that the product of two infinite series is not always equal to the product of their sums but only under a certain condition, the discussion of which is beyond the scope of this elementary treatise and here this condition is fulfilled since x is numerically less than unity.

Cor. (i) *In the expansion of* $(1-x)$.

$$T_{r+1} = (-1)^r \cdot \frac{n(n-1)(n-2)\cdots(n-r+1)}{\lfloor r} x^r.$$

(ii) *In the expansion of* $(1+x)^{-n}$.

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\cdots(-n-r+1)}{\lfloor r} x^r$$

$$= (-1)^r \cdot \frac{n(n+1)(n+2)\cdots(n+r-1)}{\lfloor r} x^r.$$

(iii) *In the expansion of* $(1-x)^{-n}$.

$$T_{r+1} = \frac{-n(-n-1)(-n-2)\cdots(-n-r+1)}{\lfloor r}(-x)^r$$

$$= (-1)^r \cdot \frac{n(n+1)(n+2)\cdots(n+r-1)}{\lfloor r} \cdot (-1)^r \cdot x^r$$

$$= \frac{n(n+1)(n+2)\cdots(n+r-1)}{\lfloor r} x^r,$$

$$[\because (-1)^r \cdot (-1)^r = (-1)^{2r} = 1.]$$

Note. From (iii), it is clear that *every term in the expansion of* $(1-x)^{-n}$ *is positive.*

85. Particular cases.

The students should be familiar with the following particular cases of the Binomial theorem as they occur very frequently.

(1) $(1-x)^n = 1 - nx + \frac{n(n-1)}{\lfloor 2} x^2 - \cdots\cdots$

(2) $(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{\lfloor 2} x^2 - \cdots\cdots$

(3) $(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{\lfloor 2} x^2 + \cdots\cdots$

(4) $(1+x)^{-1} = 1 - x + x^2 - \cdots + (-1)^r x^r + \cdots\cdots$

(5) $(1-x)^{-1} = 1 + x + x^2 + \cdots + x^r + \cdots\cdots$

(6) $(1-x)^{-2} = 1 + 2x + 3x^2 + \cdots\cdots + (r+1)\,x^r + \cdots$

(7) $(1-x)^{-3} = 1 + 3x + 6x^2 + \cdots\cdots + \frac{1}{2}\,(r+1)(r+2)\,x^r + \cdots$

(8) $(1-x^2)^{-\frac{1}{2}} = 1 + \frac{1}{2}x^2 + \frac{1.3}{2.4}\,x^4 + \cdots$

$$+ \frac{1.3.5\cdots(2r-1)}{2.4.6\cdots(2r)}\,x^{2r} + \cdots$$

86.   Expansions of $(a+x)^n$, *n being negative or fractional.*

When n is not a positive integer, we must write

$$(a+x)^n \text{ in the form } \left\{a\left(1+\frac{x}{a}\right)\right\}^n, \text{ or, } \left\{x\left(1+\frac{a}{x}\right)\right\}^n,$$

according as $x < a$ or $> a$ and then expand.   Then,

(i) Let $x < a$; then $x/a < 1$.

$$\therefore \quad (a+x)^n = \left\{a\left(1+\frac{x}{a}\right)\right\}^n = a^n\left(1+\frac{x}{a}\right)^n$$

$$= a^n\left\{1 + n\cdot\frac{x}{a} + \frac{n(n-1)}{\lfloor 2}\left(\frac{x}{a}\right)^2 + \cdots\right\}$$

$$= a^n + n\cdot a^{n-1}x + \frac{n(n-1)}{\lfloor 2}\,a^{n-2}x^2 + \cdots$$

(ii) Let $x > a$, then $a/x < 1$.

$$\therefore \quad (a+x)^n = \left\{x\left(1+\frac{a}{x}\right)\right\}^n = x^n\left(1+\frac{a}{x}\right)^n$$

$$= x^n\left\{ \ + n\cdot\frac{a}{x} + \frac{n(n-1)}{\lfloor 2}\cdot\left(\frac{a}{x}\right)^2 + \cdots\right\}$$

$$= x^n + n\cdot x^{n-1}a + \frac{n(n-1)}{\lfloor 2}\,x^{n-2}a^2 + \cdots$$

87.   Greatest Term.

The general discussion of the method of finding the greatest term is not quite necessary as the student will have no difficulty in determining the greatest term in any particular case, the method of obtaining it being the same as explained in Art. 77.   This is illustrated in the examples below.

88. Illustrative Examples.

Ex. 1. *Expand to four terms* $(1-x)^{-3}$, *given* $|x| < 1$.

By the formula of Art. 81, since $|x| < 1$, we have

$$(1-x)^{-3} = 1 + (-3)(-x) + \frac{(-3)(-3-1)}{2!}(-x)^2$$
$$+ \frac{(-3)(-3-1)(-3-2)}{3!}(-x)^3 + \cdots$$

$$= 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \cdots$$

$$= 1 + 3x + 6x^2 + 10x^3 + \cdots$$

Ex. 2. *Expand* $\dfrac{1}{\sqrt{1+x^2}}$, *given* $x > 1$.

$$\frac{1}{\sqrt{1+x^2}} = (1+x^2)^{-\frac{1}{2}} = \left\{ x^2\left(1 + \frac{1}{x^2}\right)\right\}^{-\frac{1}{2}} = x^{-1}\left(1 + \frac{1}{x^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{x}\left(1 + \frac{1}{x^2}\right)^{-\frac{1}{2}}$$

$$= \frac{1}{x}\left\{ -\frac{1}{2}\cdot\frac{1}{x^2} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)}{1.2}\cdot\frac{1}{x^4} + \frac{-\frac{1}{2}(-\frac{1}{2}-1)(-\frac{1}{2}-2)}{1.2.3}\cdot\frac{1}{x^6} + \cdots\right\}$$

$$= \frac{1}{x} - \frac{1}{2}\cdot\frac{1}{x^3} + \frac{1.3}{2.4}\cdot\frac{1}{x^5} - \frac{1.3.6}{2.4.6}\frac{1}{x^7} + \cdots$$

Ex. 3. *Write down the coefficient of the* $(r+1)th$ *term in the expansion of* $(1-2x)^{-\frac{1}{2}}$. 　　　　　　　　　[*C. U. 1938*]

By the formula of Art. 84,

$$t_{r+1} = \frac{\frac{1}{2}\cdot(\frac{1}{2}+1)(\frac{1}{2}+2)\cdots(\frac{1}{2}+r-1)}{\lfloor r}(2x)^r$$

$$= \frac{1.3.5\cdots(2r-1)}{2^r.\lfloor r}\cdot 2^r.x^r$$

$$= \frac{1.3.5\cdots(2r-1)}{\lfloor r}\cdot x^r,$$

$\therefore$ The coeff. of $x^r = \dfrac{1.3.5\cdots(2r-1)}{\lfloor r}$.

Note. It can be put in the form $\dfrac{1.3.5\cdots(2r-1).2.4.6\cdots 2r}{\lfloor r.2.4.6\cdots 2r}$

$$= \frac{\lfloor r}{\lfloor r.\ 2^r.\lfloor r} = \frac{\lfloor 2r}{\{\lfloor r\}^2.2^r}.$$

Ex. 4. *Find the coefficient of x^5 in the expansion of*
$$(1+2x+3x^2+\cdots \cdots \text{ to infinity})^{\frac{3}{2}}.$$

Now, $(1+2x+3x^2+\cdots\cdots)^{\frac{3}{2}}=\{(1-x)^{-2}\}^{\frac{3}{2}}=(1-x)^{-3}.$

The $(r+1)$th term of $(1-x)^{-3}=\frac{1}{2}\,(r+1)(r+2)\,x^r.$ [*Art. 85*]

∴ The coefficient of $x^5=\frac{1}{2}\,(5+1)(5+2)=21.$

Ex. 5. *If $y=x-x^2+x^3-x^4+\cdots\cdots$ to infinity, show that*
$$x=y+y^2+y^3+\cdots\cdots \text{ to infinity}.$$

Since $y=x-x^2+x^3-x^4+\cdots\cdots$

∴ $1-y=1-x+x^2-x^3+x^4\cdots\cdots$
$$=(1+x)^{-1}=\frac{1}{1+x}.$$

∴ $1+x=\dfrac{1}{1-y}=(1-y)^{-1}=1+y+y^2+y^3+y^4+\cdots\cdots$

∴ $x=y+y^2+y^3+y^4+\cdots\cdots$

Ex. 6. *Find the coefficient of x^{10} in $\dfrac{1+x}{(1-x)^3}$.*   [*C. U. 1937*]

$$\frac{1+x}{(1-x)^3}=(1+x)(1-x)^{-3}=(1-x)^{-3}+x(1-x)^{-3}.$$

∴ The reqd. coefficient of x^{10}
$$=\text{coeff. of }x^{10}\text{ in }(1-x)^{-3}+\text{coeff. of }x^9\text{ in }(1-x)^{-3}$$
$$=\tfrac{1}{2}.11.12+\tfrac{1}{2}.10.11=66+55=121.$$

Ex. 7. *Show that the 11th term in the expansion of $(1+3x)^{\frac{17}{2}}$ is the first which has a negative sign.*

Let $(r+1)$th term be the first negative term.

Now, $T_{r+1}=T_r\times\dfrac{n-r+1}{r}\cdot 3x=T_r\times\dfrac{\frac{17}{2}-r+1}{r}\cdot 3x$
$$=T_r\times\frac{9\frac{1}{2}+r}{r}\cdot 3x.$$

Since x and T_r are positive, T_{r+1} will be the *first* negative term as soon as $9\frac{1}{2}-r$ is negative, *i.e.*, as soon as the integer $r>9\frac{1}{2}$ *i.e.*, where $r=10$.

∴ The $(r+1)$th *i.e.*, the 11th term is the first negative term.

Otherwise :

The $(r+1)$th term in the expansion of $(1+x)^n$ being

$$\frac{n(n-1)\cdots(n-r+1)}{r}x^r,$$

it is evident that the terms will continue to be positive, so long as $n+1 > r$, *i.e.*, so long as $r < n+1$.

Hence, in the present case, the terms will continue to be positive, so long as $r < 8\frac{1}{2}+1$, *i.e.*, $r < 9\frac{1}{2}$.

$\therefore$ The first negative term will occur when the integer r is $> 9\frac{1}{2}$, *i.e.*, when $r=10$. $\therefore$ 11th term is the first negative term.

Ex. 8. *Find the greatest term in the expansion of* $(1-x)^{-n}$, *when* $x=\frac{12}{13}$ *and* $n=\frac{4}{3}$.

We have, $\qquad \dfrac{T_{r+1}}{T_r} = \dfrac{n+r-1}{r} \cdot x$ numerically

$$= \frac{\frac{4}{3}+r-1}{r}\cdot\frac{12}{13} = \frac{3r+1}{3r}\cdot\frac{12}{13}.$$

$\therefore \qquad T_{r+1} > T_r$ so long as $12(3r+1) > 13.3r$,

i.e., $36r+12 > 39r$,

i.e., $r < 4$.

Hence, for all values of r up to 3, we have $T_{r+1} < T_r$, but if $r=4$ then $T_{r+1} = T_r$ and these are the greatest terms. Thus the 4th and 5th terms are equal to one another and the greatest.

Examples X(C)

1. Expand up to four terms :

(i) $(1-2x)^{-\frac{1}{2}}$.   (ii) $(1-x^2)^{-\frac{2}{3}}$.   (iii) $(1-2x)^{\frac{3}{4}}$

(iv) $\dfrac{x}{1+x^2}$.   (v) $\dfrac{x}{\sqrt{(a^2-x^2)}}$.   (vi) $\dfrac{1}{\sqrt[3]{(1-3x)^2}}$.

(vii) $(a+x)^{-2}$.   (viii) $(x+x^2)^{-4}$.

2.   Prove that the coefficient of x^n in $(1-x)^{-(m+1)}$ is equal to that of x^m in $(1-x)^{-(n+1)}$.

3.   Write down the coefficient of x^r in $(1-2x)^{-1}$.

4.   Find the general term in the expansion of $(1-nx)^{-\frac{1}{n}}$.

5. (i) Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\dfrac{\lfloor 2r}{\{\lfloor r\}^2}$.

(ii) If t_r denote the middle term of $(1+x)^{2r}$, then will $t_0 + t_1 + t_2 + \cdots = (1-4x)^{-\frac{1}{2}}$.

6.   Show that the general term in the expansion of $(1+y)^{-\frac{p}{q}}$ is $\dfrac{p(p+q)(p+2q)\cdots\{p+(r-1)q\}}{1.2.3\cdots\cdots r.q^r}x^r$.

7.   What term of $(1-x)^{-\frac{1}{2}}$ is $\frac{1}{13}$ of the same term of $(1+x)^{-\frac{1}{2}}$?

8.   Show that three consecutive terms in the expansion of $(1+x)^n$ can be in continued proportion if $n+1=0$.

9.   Expand in ascending powers of x as far as x^3,
$$\sqrt{(1+x+x^2+x^3+\cdots \text{ to } \infty)}.$$

10.   Find the coefficient of x^r in the expansion of
$$(1-x+x^2-x^3+\cdots \text{ to } \infty)^3.$$

11.   Prove that
$$(1-x+x^2-\cdots \text{ to } \infty) \times (1+x+x^2+\cdots \text{ to } \infty)$$
$$= 1+x^2+x^4+\cdots \text{ to } \infty.$$

12. Expand $\sqrt{\left(\dfrac{1+x}{1-x}\right)}$ in ascending powers of x as far as x^4.

13. Show that
$$(1 + x + x^2 + \cdots \text{ to } \infty)^2 = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$$
[*C. U. 1922*]

14. (i) Show that the coefficient of x^5 in $\dfrac{1+x}{1-x}$ is 2.

    (ii) Find the coefficient of x^n in $\dfrac{(1+x)^2}{(1-x)^2}$.

[*C. U. 1939*]

15. Find the coefficient of x^r in the expansion of
$$(1 + 2x + 3x^2 + 4x^3 + \cdots \text{ to } \infty)^2.$$

16. Find the coefficient of x^n in the expansion of
$$(1 - 2x + 3x^2 - 4x^3 + \cdots \text{ to } \infty)^{-n}.$$

17. Prove that the coefficient of x^4 in the expansion of
$$(1 - x + x^2 - x^3)^{-1} \text{ is } 1.$$
[Write $(1 - x + x^2 - x^3)^{-1} = (1 + x)(1 - x^4)^{-1}$]

18. Find the coefficient of x^n in the expansion of

  (i) $\dfrac{(1+x)^2}{(1-x)^3}$.   (ii) $\dfrac{1 + 4x^2 + x^4}{(1-x)^4}$.

  (iii) $\dfrac{x}{(1-2x)(1-3x)}$.

19. Find the coefficient of x^r in
$$(1 + x)(1 + x^2)(1 + x^4)(1 + x^8)\cdots \text{ to } \infty, \ x^2 < 1.$$

20. Prove that the coefficient of x^r in the expansion of $\dfrac{1}{1 + x + x^2}$ is 1, 0 or -1, according as r is of the form $3n$, $3n - 1$, or $3n + 1$.

21. Expand $(1 - x + x^2)^{-\frac{1}{2}}$ in ascending powers of x as far as x^3.

22. Find the first three terms in the expansion of

$$\frac{1}{(1 + x)^2 \sqrt{1 - x}}.$$

23. Prove that

$$(1 + x + x^2 + \cdots \text{ to } \infty \ )(1 + 2x + 3x^2 + \cdots \text{ to } \infty \ )$$
$$= \tfrac{1}{2} (1.2 + 2.3x + 3.4x^2 + \cdots \text{ to } \infty \ ).$$

24. If $y = 2x + 3x^2 + 4x^3 + \cdots$ to ∞, express x in a series of ascending powers of y, up to the third term.

$$[\ C. \ U. \ 1931 \ ]$$

25. If $y = 3x + 6x^2 + 10x^3 + \cdots$ to ∞, then

$$x = \frac{y}{3} - \frac{1.4}{3^2.\lfloor 2} y^2 + \frac{1.4.7}{3^3.\lfloor 3} y^3 - \cdots\cdots$$

26. What is the first negative term in the expansion of $(1 + x)^{2\frac{1}{2}}$ and what is its coefficient ?

27. Find how many terms are positive in the expansion of $(1 - x)^{2\frac{2}{5}}$.

28. Find which are the greatest terms in the expansion of

(i) $(1 - x)^{-3}$, when $x = \tfrac{3}{4}$.

(iii) $(1 + x)^{6\frac{1}{2}}$, when $x = \tfrac{2}{3}$.

29. Show that

$$x^n = 1 + n\left(1 - \frac{1}{x}\right) + \frac{n(n + 1)}{2 \ !}\left(1 - \frac{1}{x}\right)^2 + \cdots\cdots \text{ to } \infty.$$

30. Prove that

$$(1 + x)^2 = 1 + \frac{2x}{1 + x} + \frac{3x^2}{(1 + x)^2} + \frac{4x^3}{(1 + x)^3} + \cdots$$

ANSWERS

1. (i) $1 + x + \frac{3}{2}x^2 + \frac{5}{2}x^3 + \frac{3.5}{8}x^4 + \cdots\cdots$

(ii) $1 + \frac{2}{3}x^2 + \frac{5}{9}x^4 + \frac{40}{81}x^6 + \frac{110}{243}x^8 + \cdots$

(iii) $1 - \frac{3}{2}x - \frac{3}{8}x^2 - \frac{5}{16}x^3 - \frac{45}{128}x^4 - \cdots$

(iv) $x - x^3 + x^5 - x^7 + x^9 - \cdots$

(v) $\frac{x}{a} + \frac{1}{2}\frac{x^3}{a^3} + \frac{3}{8}\frac{x^5}{a^5} + \frac{5}{16}\frac{x^7}{a^7} + \cdots$

(vi) $1 + 2x + 5x^2 + \frac{40}{3}x^3 + \cdots$

(vii) $\frac{1}{a^2} - \frac{2x}{a^3} + 3\frac{x^2}{a^4} - 4\frac{x^3}{a^5} + \cdots\cdots$

(viii) $x^{-4} - 4x^{-3} + 10x^{-2} - 20x^{-1} + \cdots$

3. 2^r.   **4.** $\dfrac{(n+1)(2n+1)(3n+1)\cdots\cdots\{(r-1)\,n+1\}}{r\,!}x^r$.

7. 8th term.   **9.** $1 + \frac{1}{2}x + \frac{3}{8}x^2 + \frac{5}{16}x^3$.   **10.** $(-1)^r.\frac{1}{2}(r+1)(r+2)$.

12. $1 + x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{3}{8}x^4$.   **14.** (ii) $4n$.   **15.** $\frac{1}{6}(r+1)(r+2)(r+3)$.

16. $\lfloor 2n / \{\lfloor n \}^2$.   **18.** (i) $2n^2 + 2n + 1$.   (ii) $n^3 + 3n$.   (iii) $3^n - 2^n$.   **19.** 1.

21. $1 + \frac{1}{2}x - \frac{1}{8}x^2 - \frac{7}{16}x^3 - \cdots\cdots$ **22.** $1 - \frac{3}{2}x + \frac{19}{8}x^2 - \cdots\cdots$

24. $x = \frac{1}{2}y - \frac{3}{8}y^2 + \frac{5}{16}y^3 - \cdots\cdots$ **26.** 5th ; $-\frac{5}{128}$.

27. 2 terms.   **28.** (i) 6th and 7th terms.   (ii) 3rd and 4th.

89.   Application of the Binomial Theorem.

The Binomial Theorem has many important applications, the most useful of which are the determination of approximate values of certain algebraical and arithmetical quantities, the summation of certain infinite series and the expansion of some rational fractions. These are illustrated in the examples below.

90.   Illustrative Examples.

Ex. 1.   *Apply the Binomial Theorem to find to 4 places of decimals the value of* $\sqrt{(102)}$.

$$\sqrt{(102)} = (100+2)^{\frac{1}{2}} = \{100\,(1+\tfrac{2}{100})\}^{\frac{1}{2}}$$

$$= 10\,(1+\tfrac{2}{100})^{\frac{1}{2}}$$

$$= 10\left\{1 + \frac{1}{2}\cdot\frac{2}{100} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{1.2}\left(\frac{2}{100}\right)^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{1.2.3}\cdot\left(\frac{2}{100}\right)^3 + \cdots\right.$$

$$= 10\,\{1 - \cdot01 - \cdot00005 + \cdot0000005 - \cdots\}$$

$$= 10\cdot0995.$$

Ex. 2. *If c be a quantity so small that c^4 may be neglected in comparison with l^4, show that*

$$\sqrt{\left(\frac{l}{l+a}\right)} + \sqrt{\left(\frac{l}{l-c}\right)} \text{ is very nearly equal to } 2 + \frac{3}{4}\cdot\frac{c^2}{l^2}.$$

The exp. $= \left(\frac{l}{l+c}\right)^{\frac{1}{2}} + \left(\frac{l}{l-c}\right)^{\frac{1}{2}} = \left(\frac{l+c}{l}\right)^{-\frac{1}{2}} + \left(\frac{l-c}{l}\right)^{-\frac{1}{2}}$

$$= \left(1+\frac{c}{l}\right)^{-\frac{1}{2}} + \left(1-\frac{c}{l}\right)^{-\frac{1}{2}}$$

$$= 1 - \frac{1}{2}\cdot\frac{c}{l} + \frac{-\frac{1}{2}\times-\frac{3}{2}}{1.2}\cdot\frac{c^2}{l^2} + \frac{-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}}{1.2.3.}\cdot\frac{c^3}{l^3}\cdots\cdots$$

$$+ 1 + \frac{1}{2}\cdot\frac{c}{l} + \frac{-\frac{1}{2}\times-\frac{3}{2}\times-\frac{5}{2}}{1.2.3}\cdot\frac{c^3}{l^3}\cdots$$

$$= 2 + \frac{3}{4}\frac{c^2}{l^2} \text{ nearly.}$$

Ex. 3. *Show that*

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = 1 + \frac{x}{1+x} + \frac{3}{2}\left(\frac{x}{1+x}\right)^2 + \frac{5}{2}\left(\frac{x}{1+x}\right)^3 + \cdots$$

$$\sqrt{\left(\frac{1+x}{1-x}\right)} = \left(\frac{1+x}{1-x}\right)^{\frac{1}{2}} = \left(\frac{1-x}{1+x}\right)^{-\frac{1}{2}}$$

$$= \left\{1 - \frac{2x}{1+x}\right\}^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}\cdot\frac{2x}{1+x} + \frac{\frac{1}{2}\cdot\frac{3}{2}}{1.2}\cdot\left(\frac{2x}{1+x}\right)^2 + \cdots$$

$$= \text{the given series, on simplification.}$$

Ex. 4. *Find the binomial expression whose expansion is*

$$1 + \frac{1}{4} + \frac{1.3}{4.8} + \frac{1.3.5}{4.8.12} + \cdots \qquad [\,C.\ U.\ 1950\,]$$

and hence determine its sum.

The series may be written in the form

$$1 + \frac{1}{4} + \frac{1.3}{2\,!} \cdot \frac{1}{4^2} + \frac{1.3.5}{3\,!} \cdot \frac{1}{4^3} + \cdots$$

$$i.e.,\ \ 1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{\frac{1}{2} \cdot \frac{3}{2}}{2\,!}\left(\frac{1}{2}\right)^2 + \frac{\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{5}{2}}{3\,!} \cdot \left(\frac{1}{2}\right)^3 \cdots$$

$$= 1 + \frac{1}{2} \cdot \frac{1}{2} + \frac{\frac{1}{2} \cdot (\frac{1}{2}+1)}{2\,!}\left(\frac{1}{2}\right)^2 + \frac{\frac{1}{2} \cdot (\frac{1}{2}+1)(\frac{1}{2}+2)}{3\,!} \cdot \left(\frac{1}{2}\right)^3 + \cdots$$

$$= (1 - \tfrac{1}{2})^{-\frac{1}{2}} = (\tfrac{1}{2})^{-\frac{1}{2}} = 2^{\frac{1}{2}} = \sqrt{2}.$$

Note. If we expand $(1-x)^{-\frac{p}{q}}$ by the Binomial Theorem and simplify, we get

$$1 + p \cdot \frac{x}{q} + \frac{p\,(p+q)}{2\,!} \cdot \frac{x^2}{q^2} + \frac{p(p+q)(p+2q)}{3\,!} \cdot \frac{x^3}{q^3} + \cdots$$

Hence we find that the *successive factors in the numerators of the coefficients are in A. P. with a common difference q, the denominator of the index*. Hence, if we have a series in which this law holds, we infer that it is a binomial expansion of the above form.

Ex. 5. *Find the sum of the coefficients of first $(r+1)$ terms in the expression of $(1-x)^{-5}$.*

Suppose $(1-x)^{-5} = a_0 + a_1 x + a_2 x^2 + \cdots + a_r\, x^r + \cdots$ $\qquad \cdots$ (1)

It is required to find $a_0 + a_1 + \cdots + a_r$.

We have $(1-x)^{-1} = 1 + x + x^2 + \cdots + x^r + \cdots$ $\qquad \cdots$ (2)

Hence, $a_0 + a_1 + \cdots + a_r$.

$\quad = $ coefficient of x^r in the product of the series (1) and (2)

$\quad = $ coefficient of x^r in $(1-x)^{-5} \cdot (1-x)^{-1}$

$\quad = $ coefficient of x^r in $(1-x)^{-6}$

$\quad = \dfrac{6.7.8 \cdots (6+r-1)}{r\,!} \qquad [\ \text{See } Art.\ 84,\ Cor.\ ]$

$\quad = \dfrac{6.7.8 \cdots (r+5)}{1.2.3 \cdots r} = \dfrac{(r+1)(r+2)(r+3)(r+4)(r+5)}{1.2.3.4.5}.$

Examples X(D)

1. Evaluate correct to three places of decimals by the Binomial Theorem :

   (i) $\sqrt[3]{(999)}$.
           (ii) $\sqrt{(1\cdot01)}$.

   (iii) $\sqrt[3]{(\frac{96}{101})}$.
          (iv) $\frac{213}{998}$.

2. If x be so small that its cube and higher powers may be neglected, show that

   (i) $\dfrac{1}{(1+4x)^3} - \dfrac{1}{(1+3x)^4} = 6x^2$.

   (ii) $\dfrac{(1+x)^{\frac{1}{2}} + (1-x)^{-\frac{1}{2}}}{(1-x^2)^{\frac{1}{2}}} = 2 + x + \frac{5}{4}x^2$.

3. If x is so large that $\dfrac{1}{x^5}$ is negligible, show that

$$\sqrt{(x^2+1)} - \sqrt{(x^2-1)} = \frac{1}{x} \text{ approximately.}$$

4. Show that $(a+x)^n : a^n$ is nearly $a+nx : a$, if x is very small compared with a.

5. If ac is small compared with b^2, show that

$$-\frac{c}{b} - \frac{ac^2}{b^3} \text{ and } -\frac{b}{a} + \frac{c}{b} + \frac{ac^2}{b^3}$$

are approximate values of the roots of $ax^2 + bx + c = 0$.

6. Show that

$$(1+x)^n = 2n\left\{1 - n\cdot\frac{1-x}{1+x} + \frac{n(n+1)}{1.2}\left(\frac{1-x}{1+x}\right)^2 - \cdots\right\}.$$

7. Show that

$$\sqrt{\frac{2}{3}} = 1 - \frac{1}{4} + \frac{1.3}{4.8} - \frac{1.3.5}{4.8.12} + \cdots\cdots$$

$$\left[\ \sqrt{\tfrac{2}{3}} = (1+\tfrac{1}{2})^{-\frac{1}{2}}\ \right]$$

8. Show that

$$\sqrt{\left(\frac{a-x}{a+x}\right)} = 1 - \frac{x}{a-x} + \frac{3}{2}\left(\frac{x}{a-x}\right)^2 - \frac{5}{2}\left(\frac{x}{a-x}\right)^3 + \cdots$$

9. Show that

$$\sqrt{8} = 1 + \frac{3}{4} + \frac{3.5}{4.8} + \frac{3.5.7}{4.8.12} + \cdots\cdots$$

10. Show that

$$\frac{x}{\sqrt{(x+1)}} = \frac{x}{1+x} + \frac{1}{2}\left(\frac{x}{1+x}\right)^2 + \frac{1.3}{2.4}\left(\frac{x}{1+x}\right)^3 + \cdots$$

11. Show that

$$1 - \frac{n+x}{1+x} + \frac{(n+2x)(n-1)}{\underline{2}.(1+x)^2} - \frac{(n+3x)(n-1)(n-2)}{\underline{3}.(1+x)^3} + \cdots = 0.$$

12. Prove that

$$1 + m \cdot \frac{2m}{1-m} + \frac{m(m+1)}{1.2}\left(\frac{2m}{1+m}\right)^2 + \cdots\cdots$$

$$= 1 + m\frac{2m}{1-m} + \frac{m(m-1)}{1.2}\left(\frac{2m}{1-m}\right)^2 + \cdots\cdots$$

13. Prove that

$$1 + \frac{2n}{3} + \frac{2n(2n+2)}{3.6} + \frac{2n(2n+2)(2n+4)}{3.6.9} + \cdots\cdots$$

$$= 2^n\left\{1 + \frac{n}{3} + \frac{n(n+1)}{3.6} + \frac{n(n+1)(n+2)}{3.6.9} + \cdots\cdots\right.$$

14. Show that

$$1 + \frac{1}{2}n + \frac{n(n+1)}{1.2}\cdot\frac{1}{2^2} + \frac{n(n+1)(n+2)}{1.2.3}\cdot\frac{1}{2^3} + \cdots = 2^n.$$

15. Prove that

$$4^m\left\{1 + \frac{m}{2} + \frac{m(m+1)}{2.4} + \frac{m(m+1)(m+2)}{2.4.6} + \cdots\right\}$$

$$= 7^m\left[1 + \frac{m}{7} + \frac{m(m-1)}{7.14} + \frac{m(m-1)(m-2)}{7.14.21} + \cdots\right].$$

16. Show that the sum of the coefficients of the first $(r+1)$ terms in the expansion of $(1-x)^{-n}$ is

$$\frac{(n+1)(n+2)\cdots(n+r)}{\lfloor r}.$$

17. Find the sum of the series

(i) $1 + \dfrac{1}{4} + \dfrac{1.4}{4.8} + \dfrac{1.4.7}{4.8.12} + \cdots\cdots$ to ∞.

(ii) $1 + \dfrac{1}{6} + \dfrac{1.3}{1.2}\dfrac{1}{6^2} + \dfrac{1.3.5}{1.2.3}\cdot\dfrac{1}{6^3} + \cdots\cdots$ to ∞.

18. Find the cube of

$$1 + \frac{1}{3}x + \frac{1.4}{3.6}x^2 + \frac{1.4.7}{3.6.9}x^3 + \cdots\cdots \text{ to } \infty,$$

as far as x^4 in its simplest form.

19. If n is a positive integer, prove that

$$1 - \frac{n^2}{1^2} + \frac{n^2(n^2-1^2)}{1^2.2^2} - \frac{n^2(n^2-1^2)(n^2-2^2)}{1^2.2^2.3^2} + \cdots = 0,$$

where the series extends up to $(n+1)$ terms.

$$\left[\text{ The series} = \text{coefficient of } x^0 \text{ in } (1+x)^n \times \left(1 + \frac{1}{x}\right)^{-n} \right]$$

ANSWERS

1. (i) 9·997. (ii) 1·005. (iii) 0·983. (iv) ·213.

17. (i) $2^{\frac{2}{3}}$. (ii) $\sqrt{\frac{3}{2}}$. **18.** $1 + x + x^2 + x^3 + x^4 + \cdots$

CHAPTER XI

LOGARITHMS

91. Definition of Logarithms.

Logarithms of a number with respect to a given base is the *index of the power* to which the base is to be raised in order to be equal to the given number.

Thus if $a^x = N$, then 'x' is the index of the power to which 'a' (which is called the base) is raised to give 'N'. Hence by definition, 'x' is the logarithm of 'N' with respect to the base 'a', and it is usually written as $x = log_a N$.

Since $2^3 = 8$, $\therefore$ $\log_2 8 = 3$, *i.e.*, 3 is the power to which 2 is to be raised to give 8. Again, since $3^4 = 81$, $\therefore$ $\log_3 81 = 4$.

It should be noted that the logarithm of the same number with respect to different bases, will be different; for example, to get the same number 64, we must raise 2 to the power 6, whereas we are to raise 4 to the power 3 and 8 to the power 2 only; hence $\log_2 64 = 6$, $\log_4 64 = 3$, $\log_4 64 = 2$.

Thus so long as the base is not stated, logarithm of a number has no meaning.

Very often when it is known which base is being used, the base is omitted for the sake of convenience. Also sometimes when any relation is found to be true whatever be the base, the base is omitted for the sake of convenience.

Note. Since the equation $a^x = -n$ cannot be satisfied by any real value of x, whether positive, or negative (a being, a real positive quantity) therefore

logarithm of a negative quantity (in a system of logarithm whose base is a real positive quantity) *must be imaginary.*

92.   Special Cases.

We know that if a be any real non-zero finite quantity, then $a^0 = 1$.

Hence, $\log_a 1 = 0$, in other words,

(i) *logarithm of unity with respect to any finite quantity (other than zero), as base, is zero.*

Again, a being any quantity, $a^1 = a$.

Hence, $1 = \log_a a$.   In other words,

(ii) *logarithm of any number with respect to itself as base is unity.*

Note. $a^x \to 0$, when $x \to -\infty$ if $a > 1$ and when $x \to +\infty$, if $0 < a < 1$.

Thus, we have $\log_a 0 \to \mp \infty$ according as $a >$ or < 1, $a > 0$. Hence,

logarithm of zero to a base greater than unity tends to minus infinity and to a base less than unity tends to plus infinity.

93.   Properties of logarithms.

(i) $\log_a (m \times n) = \log_a m + \log_a n$

in other words, *logarithm of the product of two quantities is equal to the sum of their logarithms taken separately.*

Put $\log_a m = x$, $\log_a n = y$,

and $\log_a (m \times n) = z$;

then from the definition,

$$a^x = m, \ a^y = n$$

and $a^z = m \times n = a^x \times a^y = a^{x+y}$,

so that $z = x + y$.

$\therefore$ $\log_a (mn) = \log_a m + \log_a n$.

Cor. $\log_a (m.n.p...)$

$$= \log_a m + \log_a n + \log_a p + \cdots\cdots$$

$\therefore \log_a (mnp) = \log_a (mn) + \log_a p = \log_a m + \log_a n + \log_a p$, and so on for number of factors.

(ii) $\log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$

in other words, *logarithm of the quotient of two numbers is equal to the difference of their logarithms.*

Put $\log_a m = x$, $\log_a n = y$, $\log_a \left(\dfrac{m}{n}\right) = z$.

Then from definition, $a^x = m$, $a^y = n$,

and $a^z = \dfrac{m}{n} = \dfrac{a^x}{a^y} = a^{x-y}$, so that $z = x - y$.

$\therefore \log_a \left(\dfrac{m}{n}\right) = \log_a m - \log_a n$.

(iii) $\log_a (m)^n = n \log_a m$

Or, *logarithm of a power of a number is the product of the power and the logarithm of the number.*

Put $\log_a m = x$ and $\log_a (m)^n = z$.

Then by definition, $a^x = m$, and $a^z = (m)^n = (a^x)^n = a^{nx}$.

$\therefore z = nx$. $\qquad \therefore \log_a (m)^n = n \log_a m$.

94. Change of base.

We shall now find out the relation between the logarithms of the same number to two different bases, whereby, if the logarithm of a number with respect to one base be given, that with respect to a different base may be obtained.

The relation is,

$$\log_a m = \log_b m \times \log_a b$$

Put $\log_a m = x$, $\log_b m = y$ and $\log_a b = z$.

Then from definition, $a^x = m$, $b^y = m$, $a^z = b$.

Hence $a^x = m = b^y = (a^z)^y = a^{yz}$.

$\therefore$ $x = yz$. $\qquad$ $\therefore$ $\log_a m = \log_b m \times \log_a b$.

Cor. 1. $\log_b a \times \log_a b = 1$ or, $\log_b a = 1/\log_a b$.

Let $\log_b a = x$. $\qquad \therefore$ $b^x = a$. $\qquad \therefore$ $b = a^{\frac{1}{x}}$,

and $\log_a b = y$, $\qquad \therefore$ $a^y = b$.

$\therefore \qquad a^{\frac{1}{x}} = a^y$. $\qquad \therefore$ $\dfrac{1}{x} = y$. $\qquad \therefore$ $xy = 1$,

i.e., $\log_b a \times \log_a b = 1$, or, $\log_b a = 1/\log_a b$.

Note. This result can also be obtained by putting $m = a$ in the result, $\log_a m = \log_b m \times \log_a b$, since $\log_a a = 1$.

Cor. 2. $\log_a m = \log_b m / \log_b a$.

Let $\log_a m = x$; then $a^x = m$.

Taking logarithms of both sides with respect to the base b, we get $x \log_b a = \log_b m$.

$\therefore$ $x = \log_b m / \log_b a$. $\qquad$ Hence the result.

Thus if logarithm of both m and a with respect to b be known, then the logarithm of m with respect to a is obtained by multiplying the former logarithm by the constant multiplier $\dfrac{1}{\log_b a}$. This multiplier which transforms the logarithms of a number from the old base to the new, is called the *modulus* of the new base.

95. Illustrative Examples.

Ex. 1. *Find the logarithm of 324 to the base* $3\sqrt{2}$.

Let x be the required logarithm ; then

$$(3\sqrt{2})^x = 324 = 3^4 . 2^2 = (3\sqrt{2})^4 . \qquad \therefore \quad x = 4.$$

$\therefore$ 4 is the required logarithm,

Ex. 2. *Show that*

$$7 \log \tfrac{10}{9} - 2 \log \tfrac{25}{24} + 3 \log \tfrac{81}{80} = \log 2. \qquad\qquad [\ C.\ U.\ 1922\ ]$$

$\log \tfrac{10}{9} = \log 10 - \log 9 = \log (5 \times 2) - \log 3^2 = \log 5 + \log 2 - 2 \log 3.$

$\log \tfrac{25}{24} = \log 25 - \log 24 = \log 5^2 - \log (3 \times 2^3) = 2 \log 5 - \log 3 - 3 \log 2.$

$\log \tfrac{81}{80} = \log 81 - \log 80 = \log 3^4 - \log (5 \times 2^4) = 4 \log 3 - \log 5 - 4 \log 2.$

$\therefore$ Left side

$$= 7 (\log 5 + \log 2 - 2 \log 3) - 2 (2 \log 5 - \log 3 - 3 \log 2)$$
$$+ 3 (4 \log 3 - \log 5 - 4 \log 2)$$

$$= (7 + 6 - 12) \log 2 + (7 - 4 - 3) \log 5 + (-14 + 2 + 12) \log 3$$

$$= \log 2.$$

Otherwise :

The left-hand expression

$$= \log (\tfrac{10}{9})^7 - \log (\tfrac{25}{24})^2 + \log (\tfrac{81}{80})^3$$

$$= \log \left\{ \frac{(\tfrac{10}{9})^7 \times (\tfrac{81}{80})^3}{(\tfrac{25}{24})^2} \right\}$$

$$= \log \left\{ \left(\frac{10}{3^2}\right)^7 \times \left(\frac{3^4}{10 \times 2^3}\right)^3 \times \left(\frac{3 \times 2^3 \times 2^2}{10^2}\right)^2 \right\}$$

$$= \log \left(\frac{10^7}{3^{14}} \times \frac{3^{12}}{10^3 \times 2^9} \times \frac{3^2 \times 2^{10}}{10^4} \right) = \log 2.$$

Ex. 3. *Show that*

$$\log_b a \times \log_c b \times \log_a c = 1. \qquad\qquad [\ C.\ U.\ 1934\ ]$$

$$\log_b a = \log_c a \times \frac{1}{\log_c b}. \qquad\qquad [\ by\ Cor.\ 2,\ Art.\ 94\ ]$$

$\therefore$ Left side $= \log_c a \times \dfrac{1}{\log_c b} \times \log_c b \times \log_a c$

$$= \log_c a \times \log_a c$$
$$= 1. \quad [\ by\ Cor.\ 1,\ Art.\ 94\ ]$$

Ex. 4. *If a series of numbers be in G. P., show that their logarithms are in A. P.*

Let the numbers be a, ar, ar^2, ar^3,..., ar^{n-1}.

Now, $\log a = \log a$.

$\log (ar) = \log a + \log r$.

$\log (ar^2) = \log a + \log r^2 = \log a + 2 \log r$.

$\log (ar^2) = \log a + \log r^3 = \log a + 3 \log r$.

...   .

$\log (ar^{n-1}) = \log a + \log r^{n-1} = \log a + (n-1) \log r$.

Obviously, $\therefore$ $\log (a)$, $\log (ar)$, $\log (ar^2)$......, $\log (ar^{n-1})$ are in A. P. of which the first term is $\log a$ and the common difference is $\log r$.

Note. Here for the sake of convenience, the base is omitted throughout.

Examples XI(A)

1. Find the logarithms of

(i) 125 to the base $5 \sqrt{5}$.   (ii) 1728 to the base $2 \sqrt{3}$.

(iii) $\cdot 1$ to the base $9 \sqrt{3}$.   (iv) 2401 to the base $\sqrt[3]{7}$.

2. What number is the base when 4 is the logarithm of 1296 ?

3. Prove that

(i) $\log_a m \times \log_b n = \log_b m \times \log_a n$.

(ii) $\log_2 \log_2 \log_2 16 = 1$.

4. If $\log_e m + \log_e n = \log_e (m + n)$, find m as a simple function of n.

5. Show that $\log_{10} 2$ lies between $\frac{1}{3}$ and $\frac{1}{4}$.

6. Simplify $\log_2 \sqrt{6} + \log_2 \sqrt{\frac{2}{3}}$.

7. Prove that $\log (1 + 2 + 3) = \log 1 + \log 2 + \log 3$.

8. Show that $\log \dfrac{b^n}{c^n} + \log \dfrac{c^n}{a^n} + \log \dfrac{a^n}{b^n} = 0$.

9. If the logarithm of x^2 to the base y^3 be equal to the logarithm of y^8 to the base x^{12}, find the value of each logarithm.

10. If $\log (x^2 y^3) = a$ and $\log \left(\dfrac{x}{y}\right) = b$, find $\log x$ and $\log y$ in terms of a and b.

11. If $a^2 + b^2 = 7ab$, show that
$$\log \{\tfrac{1}{3} (a + b)\} = \tfrac{1}{2} (\log a + \log b).$$

12. Prove that

(i) $\log a + \log a^2 + \log a^3 + \cdots + \log a^n = \tfrac{1}{2} n (n + 1) \log a.$

(ii) $\log a + \log a^3 + \log a^5 + \cdots + \log a^{2n-1} = n^2 \log a.$

13. If $\dfrac{\log x}{y - z} = \dfrac{\log y}{z - x} = \dfrac{\log z}{x - y}$, then $x^x y^y z^z = 1.$

14. Prove that

(i) $\log_a b \times \log_b c \times \log_c d \times \log_d a = 1.$

(ii) $\log_a b \times \log_b c \times \log_c d \times \cdots \times \log_q r \times \log_r a = 1.$

15. Prove that

(i) $\log \tfrac{81}{8} - 2 \log \tfrac{3}{2} + 3 \log \tfrac{2}{3} + \log \tfrac{3}{4} = 0.$

(ii) $7 \log \tfrac{15}{16} + 6 \log \tfrac{8}{3} + 5 \log \tfrac{2}{5} + \log \tfrac{32}{25} = \log 3.$

16. (i) $a^m = b^n$, show that $n \log_a x = m \log_b x.$

(ii) If $a^{3-x} b^{5x} = a^{x+5} b^{3x}$, then $x \log \left(\dfrac{b}{a}\right) = \log a.$

[*C. U. 1937*]

17. Find the value of
$$\frac{\log \sqrt{27} + \log 8 - \log \sqrt{1000}}{\log 1\cdot 2}.$$

18. Show that
$$7 \log \tfrac{16}{15} + 5 \log \tfrac{25}{24} + 3 \log \tfrac{81}{80} = \log 2.$$

[*C. U. 1936*]

19. Evaluate $\log_2 \sqrt{2 \sqrt{2 \sqrt{2 \dots}}}$ to $\infty.$

20. If x, y, z are in G. P., show that
$$\log_a x, \log_a y, \log_a z \text{ are in A. P.}$$

21. If a, b, c are in G. P., show that
$$\log_a x, \log_b x, \log_c x \text{ are in H. P.}$$

22. If $\dfrac{a(b+c-a)}{\log a} = \dfrac{b(c+a-b)}{\log b} = \dfrac{c(a+b-c)}{\log c}$,
show that $b^c c^b = c^a a^c = a^b b^a$.

23. Prove that
$$x^{\log y - \log z} \times y^{\log z - \log x} \times z^{\log x - \log y} = 1.$$

[C. U. 1939, '44]

24. $xy^{l-1} = a$, $xy^{m-1} = b$, $xy^{n-1} = c$, prove that
$$(m-n)\log a + (n-l)\log b + (l-m)\log c = 0.$$

25. If $y = a^{\frac{1}{1-\log x}}$, $z = a^{\frac{1}{1-\log y}}$, then $x = a^{\frac{1}{1-\log z}}$,
all the logarithms being calculated to the base a.

ANSWERS

1. (i) 2.   (ii) 6.   (iii) $-\frac{4}{5}$.   (iv) 12.   **2.** 6.   **6.** 1.   **9.** $\pm\frac{2}{3}$.
10. $\log x = \frac{1}{5}(a+3b)$; $\log y = \frac{1}{5}(a-2b)$.   **17.** $1\frac{1}{2}$.   **19.** 1.

96. Common Logarithms.

Any number can be used as the base of logarithms and corresponding to any such base, a system of logarithms of all numbers can be found. In practice, however, only two systems of logarithms are used *viz.*, *Napierian System* and *Common System*.

In the Napierian System called after the name of the inventor John Napier, the base is the incommensurable number e, where

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots\cdots$$

[See *Art. 102*]

an infinite series whose value correct to 5 places of decimals is 2·71828. This system is used in all theoretical investigations.

When approximate numerical calculations are made by means of logarithms, the logarithms are always to the base 10. On this account logarithms to the base 10 are called *Common logarithms*. This system was first introduced by Henry Briggs.

We shall now proceed with the consideration of the common system of logarithms and the base being understood to be 10 need not be written.

97. Characteristic of Mantissa.

From the equation $10^x = n$, it is evident that the common logarithms will not in general be integral and that they will not always be positive.

For instance, $123 > 10^2$ and $< 10^3$.

∴   log $123 = 2 + $ a fraction.

Again, $·02 > 10^{-2}$ and $< 10^{-1}$.

∴   log $·02 = -2 + $ a positive fraction.

Def. The integral part of the logarithm of a number is called the *Characteristic* and the decimal part (*i.e.*, the fractional part when expressed as a decimal) is called the *Mantissa*.

Thus, since log $648 = 2·81158$, the *characteristic of* log 648 is 2 and its *mantissa* is $·81152$.

Generally, if two numbers have the same figures arranged in the same order and therefore differ only in the position of the decimal point, then one must be the product of the other and some integral power of 10 ; and hence from Art. 93(i), the logarithms of numbers will differ by an integer. Thus, log $328·5 = $ log $3·285 + $ log $100 = 2 + $ log $3·285$.

Again, since log $2 = ·30103$, we have log $·02 = $ log $(2 \div 100)$ $= $ log $2 - $ log $100 = ·30103 - 2 = (-1·69897)$; but owing to the above property, as a matter of convenience *common logarithms are always written with the decimal part positive.* Thus log $·02$ is written not as $-1·69897$, but as $\overline{2}·30103$ (*i.e.*, $-2 + ·30103$). Here the minus sign refers only to the

integral portion of the logarithm and is written above the figure to which it refers. The characteristic and mantissa of log ˙02 are respectively − 2 and ˙30103 and not − 1 and − ˙69897.

Hence, *the characteristic · of the logarithm of a number may be positive or negative but the mantissa is always positive*

98. Characteristic by Inspection.

Let us first consider logarithms of numbers greater than one.

We have,

$$10^0 = \quad 1. \qquad \therefore \quad \log 1 \quad = 0.$$
$$10^1 = \quad 10. \qquad \therefore \quad \log 10 \quad = 1.$$
$$10^2 = \quad 100. \qquad \therefore \quad \log 100 \quad = 2.$$
$$10^3 = 1000. \qquad \therefore \quad \log 1000 = 3 \text{ ; and so on.}$$

Thus, it is clear that logarithm of a number between 1 and 10 lies between 0 and 1 ; *i.e.*, the logarithm of a number containing 1 digit only in its integral part is $0 + a$ positive proper fraction, and hence, its characteristic is **0.**

The logarithm of a number between 10 and 100 lies between 1 and 2 ; *i.e.*, the logarithm of a number containing 2 digits only in its integral part is $1 + a$ positive proper fraction, and hence, its characteristic is **1.**

The logarithm of a number between 100 and 1000 lies between 2 and 3 ; *i.e.*, the logarithm of a number containing 3 digits only in its integral part is $2 + a$ positive proper fraction, and hence, its characteristic is **2.**

And generally, the logarithm of a number containing **n** digits only in its integral part is $(n − 1) + a$ positive proper fraction, and hence, its characteristic is **(n − 1).**

Thus, *the characteristic of the common logarithm of any number greater than unity is positive and one less than the number of digits in the integral part of the number.*

Now let us consider the logarithms of positive numbers less than one.

We have,

$$10^0 = 1. \qquad \qquad \therefore \quad \log 1 = 0.$$

$$10^{-1} = \frac{1}{10} = \cdot 1. \qquad \therefore \quad \log \cdot 1 = -1.$$

$$10^{-2} = \frac{1}{10^2} = \cdot 01. \qquad \therefore \quad \log \cdot 01 = -2.$$

$$10^{-3} = \frac{1}{10^3} = \cdot 001. \qquad \therefore \quad \log \cdot 001 = -3 ;$$

and so on.

Thus, the logarithm of a number between $\cdot 1$ and 1 lies between -1 and 0 ; *i.e.*, the logarithm of a decimal having **no** zero just after the decimal point is $-1 +$ a positive proper fraction and hence, its characteristic is $-\mathbf{1}$.

The logarithm of a number between $\cdot 01$ and $\cdot 1$ lies between -2 and -1 ; *i.e*, the logarithm of a decimal having **1** zero just after the decimal point (*cf.* $\cdot 06$) is $-2 +$ a positive proper fraction and hence, its characteristic is $-\mathbf{2}$.

The logarithm of a number between $\cdot 001$ and $\cdot 01$ lies between -3 and -2 ; *i.e.*, the logarithm of a decimal having **2** zeros just after the decimal point (*cf.* $\cdot 0028$) is $-3 +$ a positive proper fraction and hence, its characteristic is $-\mathbf{3}$.

And generally, the logarithm of a decimal having **n** zeros just after the decimal point is $-(n+1) +$ a positive proper fraction and hence, its characteristic $-\mathbf{(n+1)}.$

Hence, we have the following **Rule :**

Thus, *the characteristic of the common logarithm of any positive number less than unity is negative and is numerically one greater than the number of zero immediately after the decimal point of the number.*

90.   Numbers having the same Mantissæ.

The mantissæ of the logarithms of all numbers consisting of the same significant digits arranged in the same order but differing only in the position of their decimal points, are the same.

Let x be any number ; then $x \times 10^m$ or $x \div 10^n$, where m and n are positive integers, is evidently a number having the same digits as x, for multiplication or division by a power of 10 does not change the sequence of significant digits in the number.

Now, $\log (x \times 10^m) = \log x + \log 10^m = \log x + m \log 10$

$$= \log x + m. \qquad \cdots \quad (1)$$

Also, $\log (x \div 10^n) = \log x - \log 10^n = \log x - n \log 10$

$$= \log x - n. \qquad (2)$$

Thus in (1) an integer being added to and in (2) an integer being subtracted from $\log x$, the decimal part *i.e.*, the mantissa remains the same in both the cases.

Hence, we get the above result.

Illustrations :

Let us consider the logarithms of the numbers

    48130 and 48·13, having given log 4813 = 3·6824.

(i) $\log 48130 = \log (4813 \times 10) = \log 4813 + \log 10$

$$= 3·6824 + 1 = 4·6824.$$

(ii) $\log 48·13 = \log \frac{4813}{100} = \log 4813 - \log 10^2$

$$= 3·6824 - 2 = 1·6824.$$

Thus the logarithms of the numbers 4813, 48130, 48·13, which have the same significant digits arranged in the same order but which differ only in the position of their decimal points, have the same mantissæ.

100. Principles of Proportional Parts.

Suppose we find from the Table, logarithms of the two numbers 6257 and 6258, and we want to find the logarithm of 6257·6 ; how are we to proceed ?

In order to meet such cases, the 'Principle of Proportional Parts' may be used. The principle may be stated as follows :

If the value of a quantity depending on a variable quantity x be tabulated for different values of x at regular small intervals, then in most cases, for a very small change in x (which is called the argument) the corresponding small change in the tabulated quantity (called the function of the argument), is proportional to the change in x.

We shall assume the truth of this principle, for a strict proof of it, with the proper restrictions under which it is true, depends on the use of Calculus. For the tables with which we are concerned, it is true for all practical purposes.

The applications of this principle *in determining the logarithm of a number* as also *in determining the number whose logarithm is given*, is illustrated in the following examples.

101. Illustrative Examples.

Ex. 1. *Given log 63374 = 4·8019111 and log 63375 = 4·8019180. find log 63·3743 and find the number whose logarithm is $\bar{2}$·8019136.*

Here log 63374 = 4·8019111 and log 63375 = 4·8019180.

Hence for an increase of 1 in the number, the increment in the logarithm is ·0000069. (This is usually spoken as 'diff. for 1 is 69').

Therefore by the Principle of Proportional Parts, increase in the logarithm for an increase of ·3 in the number is

$$·3 \times ·0000069 = ·00000207 = ·0000021 \text{ nearly.}$$

∴ log 63374·3 = 4·8019111 + ·0000021 = 4·8019132.

∴ log 63·3743 = 1·8019132.

Again, 4·8019136 lies between 4·8019111 and 4·8019180, the difference from the former being ·0000025. Hence 4·8019136 is the logarithm of a number lying between 63374 and 63375, say 63374 + x.

Then difference for 1 being 69 (*i.e.*, ·0000069) and difference for x being 25 (*i.e.*, ·0000025), 69 : 25 : : 1 : x.

$$\therefore \quad x = \tfrac{2\,5}{6\,9} = ·36\ldots\ldots$$

Hence, log 63374·36······ = 4·8019136.

The required number whose logarithm is $\bar{2}$·8019136, having the same mantissa, must be formed of the same digits and its characteristic being -2, the number must be ·06337436…

Note. The number corresponding to a given logarithm is called its **anti-logarithm.** Thus, in the above example,

$$\text{antilog } \bar{2}·8019136 = ·06337436.$$

Ex. 2. *Find the number of digits in* 4^{15} *having given log* $2 = ·30103$.

We have $\log 4^{15} = \log 2^{30} = 30 \log 2$

$$= 30 \times ·30103 = 9·0309.$$

Hence, since the characteristic of log 4^{15} is 9, it must consist of 10 digits.

Ex. 3. *Find approximately the* 7th *root of* 3·528, *having given* *log* $2 = ·30103$,   *log* $3 = ·4771213$,   *log* $7 = ·8450980$ *and log* 1197·342 $= 3·0782184$.

Let $x = (3·528)^{\frac{1}{7}} = \left(\dfrac{7^2 \times 3^2 \times 2^3}{10^3}\right)^{\frac{1}{7}}$

then $\log x = \tfrac{1}{7} [2 \log 7 + 2 \log 3 + 3 \log 2 - 3 \log 10]$

$$= \tfrac{1}{7} [2 \times ·8450980 + 2 \times ·4771213 + 3 \times ·30103 - 3]$$

$$= ·0782184 \text{ nearly.}$$

Now, log 1197·342 = 3·0782184 ;

$\therefore$ log 1·197342 = ·0782184, having characteristic 0, but manitissa same as that of log 1197·342.

Hence $x = 1·197342$ approximately.

Ex. 4. *Find to two places of decimals the value of x from the equation*

$$6^{3-4x} \cdot 4^{x+5} = 8.$$

Given log 2 = 0·3010300, log 3 = 0·4771213. [*C. U. 1938*]

Taking logarithms of both sides, we have

$$(3 - 4x) \log 6 + (x + 5) \log 4 = \log 8,$$

$$\therefore \quad (3 - 4x) \log (2 \times 3) + (x + 5) \log 2^2 = \log 2^3.$$

$$\therefore \quad (3 - 4x)(\log 2 + \log 3) + (x + 5) \, 2 \log 2 = 3 \log 2.$$

$$\therefore \quad x(-4 \log 2 - 4 \log 3 + 2 \log 2) = -3 \log 3 - 10 \log 2.$$

$$\therefore \quad x = \frac{10 \log 2 + 3 \log 3}{2 \log 2 + 4 \log 3}$$

$$= \frac{4·4416639}{2·5105452} = 1·77 \text{ nearly.}$$

Examples XI(B)

[*Use the values :*   *log* 2 = ·30103,   *log* 3 = ·4771213, *log* 7 = ·8450980, *when required.*]

1. Given log 18·906 = 1·2765997, log 18·907 = 1·2766226 ; find log 1890·635.

2. Given log 69714 = 4·8433200, log 69715 = 4·8433262 ; find log $(·000697145)^{\frac{1}{5}}$.

3. Given log 37601 = 4·5751994, log 37602 = 4·5752109 ; find the number whose logarithm is 1·5752086.

4. Given log 3 = ·4771213, log 74008 = 4·8692787, diff. for 1 = 59 ; find $(·09)^{\frac{1}{8}}$.

5. Find the number of digits in (i) 2^{40}, (ii) 3^{11}, (iii) $(540)^9$.

6. Extract the 5th root of 84, having given

$$\log 2425805 = 6\cdot3848559.$$

7. Calculate $(\cdot0020736)^{\frac{1}{7}}$, having given that

$$\log 41369 = 4\cdot6166750.$$

8. Given $\log 898665 = 5\cdot9535977$, find the value of

$$\sqrt[3]{\left(\frac{7\cdot2 \times 6\cdot3}{62\cdot5}\right)}.$$

9. Given $\log 259569 = 5\cdot4142524$, find the value of

$$\sqrt[5]{\left\{\frac{(\cdot32)^8 \times (625)^4}{(\cdot00432)^2 \times (\cdot3125)^3 \times 25}\right\}}.$$

10. Solve the following equations using the value of $\log 2$, $\log 3$ etc. when required :

(i) $3^{2-x}\cdot4^{2x-3} = 20.$ (ii) $5^{5-3x} = 2^{x+2}.$

(iii) $2^x\cdot3^{2x} = 100.$ [*C. U. 1925*]

(iv) $2^x = 3^y.$

     $2^{y+1} = 3^{x-1}.$ [*C. U. 1942*]   (v) $18y^x - y^{2x} = 81.$ $3^x = y^2.$

(vi) $(ax)^{\log a} = (by)^{\log b},\ b^{\log x} = a^{\log y}.$

(vii) $6^{3-4x}\cdot4^{x+5} = 8.$ [*C. U. 1945*]

11. Given $\log 101 = 2\cdot0043214$

and $\log 111\cdot5675 = 2\cdot0475354$, find the value of

$$\tfrac{101}{100} + \left(\tfrac{101}{100}\right)^2 + \left(\tfrac{101}{100}\right)^3 + \cdots\cdots \text{ to 10 terms}.$$

12. If the number of persons born in any year be $\frac{1}{48}$th of the whole population at the commencement of the year and the number of those who die be $\frac{1}{60}$th of it, find in what time the population will be doubled.

[*Given* $\log 241 = 2\cdot3320170,\ \log 240 = 2\cdot3802112$]

ANSWERS

1. $3\cdot2766077$.   2. $\bar{1}\cdot3686646$.   3. $37\cdot6018$.   4. $\cdot7400827$.

5. (i) 13.   (ii) 6.   (iii) 25.   6. $2\cdot425805$.   7. $\cdot41369$.

8. $\cdot898665$.   9. $259\cdot569$.   10. (i) $x=2\cdot961$.   (ii) $x=1\cdot206$.

(iii) $x=1\cdot593$.   (iv) $x=2\cdot71, y=1\cdot71$.   (v) $x=2, y=\pm3$;

$x=-2, y=\pm\frac{1}{3}$.   (vi) $x=\dfrac{1}{a}, y=\dfrac{1}{b}$.   (vii) $1\cdot77$.   11. $10\cdot5675$.

12. $166\cdot7$ years nearly.

Examples XI(C)

[*Additional Example on Chapter XI*]

[*Use the values* : $\log 2 = \cdot30103$, $\log 3 = \cdot47712$,

$\log 7 = \cdot84510$, *when required*]

1. A Geometrical and a Harmonical progression have the same pth, qth, rth terms of a, b, c respectively ; show that $a(b-c)\log a + b(c-a)\log b + c(a-b)\log c = 0$.

2. If $\log (x+z) + \log (x-2y+z) = 2 \log (x-z)$, then x, y, z are in H. P.

3. If $\dfrac{xy \log (xy)}{x+y} = \dfrac{yz \log (yz)}{y+z} = \dfrac{zx \log (zx)}{z+x}$, then
$$x^x = y^y = z^z.$$

4. Evaluate $\sqrt[3]{\{9 \sqrt{(3 \sqrt 2)}\}}$.

[*Given log $1\cdot62683 = \cdot21134$*]

5. Find correct to 3 places of decimals the value of logarithm of 40 to the base 12.

6. Show that
$$\log_{10} 2 + 16 \log_{10} \frac{16}{15} + 12 \log_{10} \frac{25}{24} + 7 \log_{10} \frac{81}{80} = 1.$$

[*C. U. 1940*]

7.   Prove that

(i) $\log \frac{75}{16} - 2 \log \frac{5}{9} + \log \frac{32}{243} = \log 2$.

(ii) $\log \frac{11}{15} + \log \frac{490}{297} - 2 \log \frac{7}{9} = \log 2$.

8.   If $7^{3x+2} + 4^{x+2} = 7^{3x+1} + 2^{2x+6}$, find by the help of logarithmic tables the value of x.   [*C. U. 1941*]

9.   Given $\log_{10} x - \log_{10} \sqrt{x} = \dfrac{2}{\log_{10} x}$, find x.

10.   The logarithm of a certain number to a certain base is 6 and the logarithm of 8 times the number to the base formed by the product of the first base and 25 is 3 ; find the base.

11.   If $a_1^{x_1} = a_2^{x_2} = a_3^{x_3} = \cdots = a_n^{x_n}$ and if $a_1, a_2, \ldots a_n$ be in G. P., prove that $x_1, x_2, \ldots x_n$ are in H. P.

12.   Solve the following :

(i) $\sqrt{5^x} + \sqrt{5^{-x}} = 2 \cdot 9$.

(ii) $18^{8-4x} = (54 \sqrt{2})^{3x-2}$.

(iii) $4^{2x} - 8 \cdot 4^x + 12 = 0$.

(iv) $3^{x+2y} = 5 \; ; \; 3^{2x-5y} = 7$.

(v) $2 \log_{10} x = 1 + \log_{10} (x - 1\frac{3}{5})$.

13.   If $5^{5-3x} + 4^{\frac{1}{2}x+3} = 5^{7-3x} - 2^{x+2}$, find the value of x, correct to 3 places of decimals.   [*C. U. 1943*]

14.   The number of births in a certain town every year is 25 per thousand and the number of deaths is 20 per thousand of the population at the beginning of every year. The population in a certain year is 25000 ; find the population 20 years afterwards.

[*Given* $\log 40965 = 4 \cdot 6124180$; $\log 1025 = 3 \cdot 0107239$]

15. A shop-keeper has one maund of rice ; as soon as he sells four seers, he mixes with the remainder four seers of an inferior kind. How often must he repeat the process that half of the whole may be of the inferior kind.

ANSWERS

4. 1·62683.　　5. 1·485.　　8. ·03.　　9. (i) 100 or $\frac{1}{100}$.

(ii) 2 or 8.　　10. 12·5.　　12. (i) ±1·139.　　(ii) 1·294.

(iii) ·5, 1·292.　　(iv) $x = 1·2075, y = ·129.$　　(v) $x = 2, 8.$　13. 1·206.

14. 40965 nearly.　　15. 7 times nearly.

EXPONENTIAL AND LOGARITHMIC SERIES
SEC. A. EXPONENTIAL

102. The Series e.

The infinite series

$$1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \cdots\cdots + \frac{1}{\lfloor r} + \cdots\cdots$$

is usually denoted by the letter e and is of fundamental importance in mathematics.

(i) *To prove that e is finite and lies between 2 and 3.*

$$e = 1 + \frac{1}{\lfloor 1} + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots\cdots$$

$$= 2 + \frac{1}{\lfloor 2} + \frac{1}{\lfloor 3} + \cdots\cdots$$

Hence, it is clear that e is greater than 2.

Again, $\lfloor 3 = 3.2.1.$

$\therefore \qquad \lfloor 3 > 2.2.1 \ i.e., \ > 2^2.$

$\therefore \qquad \dfrac{1}{\lfloor 3} < \dfrac{1}{2^2}.$

Similarly, $\dfrac{1}{\lfloor 4} < \dfrac{1}{2^3} \; ; \; \dfrac{1}{\lfloor 5} < \dfrac{1}{2^4} \; ;$ etc.

Hence, $\qquad e = 1 + \dfrac{1}{\lfloor 1} + \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \dfrac{1}{\lfloor 4} + \cdots$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$$

$i.e., \qquad < 1 + \dfrac{1}{1 - \frac{1}{2}}, \ i.e., \ < 1 + 2 \ i.e., \ < 3.$

$\therefore \ \ e$ lies between 2 and 3.

(ii) *To prove that e is an incommensurable number.*

If possible, let e be commensurable and equal to m/n, where m and n are positive integers.

Then, $\dfrac{m}{n} = 1 + 1 + \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \cdots + \dfrac{1}{\lfloor n} + \dfrac{1}{\lfloor n+1} + \cdots$

Multiplying both sides by $\lfloor n$, we get

$$m \lfloor n-1 = \lfloor n + \lfloor n + \dfrac{\lfloor n}{\lfloor 2} + \cdots + 1 + \dfrac{1}{n+1} + \dfrac{1}{(n+1)(n+2)} + \cdots$$

Evidently $\lfloor n-1$ is an integer and all the terms on the right side are integers except $\dfrac{1}{n+1}, \dfrac{1}{(n+1)(n+2)}, \cdots$

$$\therefore \quad \dfrac{1}{n+1} + \dfrac{1}{(n+1)(n+2)} + \cdots = \text{some integer.} \quad \cdots \quad (1)$$

Each term of the left side of (1) being positive, their sum is obviously $> \dfrac{1}{n+1}$,

$$\text{and} < \dfrac{1}{n+1} + \dfrac{1}{(n+1)^2} + \dfrac{1}{(n+1)^3} + \cdots$$

$$i.e., < \dfrac{1}{n+1} \Big/ \left(1 - \dfrac{1}{n+1}\right) \ i.e., \ < \dfrac{1}{n}.$$

Thus, the left side of (1) lies between $1/(n+1)$ and $1/n$ and is therefore a proper fraction ; but it has been shown that the left side is an integer. Hence the assumption that e is commensurable is wrong. Thus e cannot be a commensurable number.

Note. e being an incommensurable number, by taking a sufficiently large number of terms of the series, its value can be calculated to any desired order of approximation. The value of e correct to 6 places of decimals is found to be 2·718282.

103.   Expansion of e^x.

To prove that for all values of x

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots + \frac{x^r}{r!} + \cdots\cdots \text{ to } \infty, \cdots \text{ (1)}$$

If $n > 1$, $\left(\text{so that } \frac{1}{n} < 1\right)$, we have by the Binomial Theorem,

$$\left(1 + \frac{1}{n}\right)^{nx} = 1 + nx \cdot \frac{1}{n} + \frac{nx(nx-1)}{\underline{2}} \cdot \frac{1}{n^2} + \cdots\cdots$$

$$= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{\underline{2}} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{\underline{3}} + \cdots$$

Hence, when n is infinitely large,

$$\left(1 + \frac{1}{n}\right)^{nx} = 1 + x + \frac{x^2}{\underline{2}} + \frac{x^3}{\underline{3}} + \cdots\cdots \quad \text{(2)}$$

Therefore, putting $x = 1$ in (2), we have when n is infinitely large,

$$\left(1 + \frac{1}{n}\right)^n = 1 + \frac{1}{\underline{1}} + \frac{1}{\underline{2}} + \frac{1}{\underline{3}} + \cdots\cdots = e.$$

Now, $\left\{\left(1 + \frac{1}{n}\right)^n\right\} = \left(1 + \frac{1}{n}\right)^{nx}$ always.

$\therefore$ Making n infinitely large, we have for all values of x,

$$e^x = 1 + \frac{x}{\underline{1}} + \frac{x^2}{\underline{2}} + \frac{x^3}{\underline{3}} + \cdots\cdots + \frac{x^r}{\underline{r}} + \cdots \text{ to } \infty.$$

Cor. 1.   Since the series (1) is true for all values of x, therefore putting $-x$ and -1 for x in the above series, we have

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^r \frac{x^r}{r!} + \cdots\cdots$$

and $e^{-1} = 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \cdots + (-1)^r \frac{1}{r!} + \cdots\cdots$

Cor. 2. We have seen above that when n becomes infinitely large, the value of $\left(1 + \dfrac{1}{n}\right)^n = e$. This is usually expressed by this notation,

$$\underset{n \to \infty}{Lt} \left(1 + \frac{1}{n}\right)^n = e \; ; \; \text{similarly,} \; \underset{n \to \infty}{Lt} \left(1 + \frac{1}{n}\right)^{nx} = e^x.$$

Note 1. The expansion of e^x is known as **Exponential Series**.

Note 2. The proof given above though short is not satisfactory. It is assumed in the course of the proof that when n is infinitely large, the value of

$$\frac{x\left(x - \dfrac{1}{n}\right)\left(x - \dfrac{2}{n}\right) \cdots \left(x - \dfrac{r-1}{n}\right)}{\underline{|r}} \quad \text{is } \frac{x^r}{\underline{|r}} \text{ for all values of } r.$$

Sometimes it is argued that when n is infinitely large.
$$1/n = 0, \; 2/n = 0, \; 3/n = 0 \ldots \ldots r/n = 0.$$

But it cannot safely be assumed that r/n is zero when both r and n become infinitely large. The above relation can, however, be proved by a more rigorous method but it is beyond the scope of the present treatise.

Note 3. The above exponential series has been established for *real values* of x. It should be noted that e^x, when x is *imaginary*, has not the same meaning as when x is real, although in text-books on Higher Algebra, it is *defined* as equivalent to the same series as when x is real.

Alternative method of Proof. [*]

$$\text{Let } f(m) \equiv 1 + \frac{m}{\underline{|1}} + \frac{m^2}{\underline{|2}} + \frac{m^3}{\underline{|3}} + \cdots + \frac{m^r}{\underline{|r}} + \cdots \qquad (3)$$

$$\text{then } f(n) \equiv 1 + \frac{n}{\underline{|1}} + \frac{n^2}{\underline{|2}} + \frac{n^3}{\underline{|3}} + \cdots + \frac{n^r}{\underline{|r}} + \cdots \qquad (4)$$

---

[*] This proof is due to Prof. Hill and it only assumes the truth of the Binomial Theorem for a positive integral exponent. [*Proc. of the Cambridge Phil. Soc. 5, p. 415.*]

[†] The series $f(m)$, $f(n)$ and $f(m+n)$ are *convergent* for all values of m and n.

and $f(m+n) \equiv 1 + \dfrac{(m+n)}{\lfloor 1} + \dfrac{(m+n)^2}{\lfloor 2} + \cdots + \dfrac{(m+n)^r}{\lfloor r} + \cdots$ (5)

Now, multiplying together the series (3) and (4) and collecting the terms of the same degree in m and n, we have

$$f(m) \times f(n) = 1 + \frac{(m+n)}{\lfloor 1} + \left(\frac{m^2}{\lfloor 2} + \frac{mn}{\lfloor 1} + \frac{n^2}{\lfloor 2}\right) + \cdots$$

$$+ \left(\frac{m^r}{\lfloor r} + \frac{m^{r-1}}{\lfloor r-1} \cdot \frac{n}{\lfloor 1} + \frac{m^{r-2}}{\lfloor r-2} \frac{n^2}{\lfloor 2} + \cdots + \frac{n^r}{\lfloor r}\right) + \cdots$$

$$= 1 + \frac{(m+n)}{\lfloor 1} + \frac{1}{\lfloor 2}(m+n)^2 + \cdots$$

$$+ \frac{1}{\lfloor r}\left\{m^r + rm^{r-1} n + \frac{r(r-1)}{\lfloor 2}m^{r-2}n^2 + \cdots + n^r\right\} + \cdots$$

$$= 1 + \frac{(m+n)}{\lfloor 1} + \frac{1}{\lfloor 2}(m+n)^2 + \cdots + \frac{1}{\lfloor r}(m+n)^r + \cdots$$

$$= f(m+n).$$

$\therefore\ f(m) \times f(n) = f(m+n)$ for all values of m and n.

Similarly,

$$f(m) \times f(n) \times f(p) = f(m+n) \times f(p) = f(m+n+p)$$

and generally,

$f(m) \times f(n) \times f(p) \times \cdots$ to a finite number of factors

$= f(m+n+p+\cdots$ to a finite number of terms)

for all values of $m,\ n,\ p,\ldots$ $\qquad \cdots \qquad \cdots$ (A)

(i) Let x be a *Positive integer*.

Putting $m = n = p = \cdots = 1$ in (A), we have

$f(1) \times f(1) \times f(1) \times \cdots$ to x factors

$= f(1 + 1 + 1 + \cdots$ to x terms).

$\therefore\ \{f(1)\}^x = f(x).$

(ii) Let x be any *positive rational fraction* h/k, where h and k are positive integers.

Putting $m = n = p = \cdots = \dfrac{h}{k}$ in (A),

$$\left(\frac{h}{k}\right) \times f\left(\frac{h}{k}\right) \times \cdots \text{ to } k \text{ factors}$$

$$= f\left(\frac{h}{k} + \frac{h}{k} + \cdots \text{ to } k \text{ terms}\right).$$

$$\therefore \quad \left\{f\left(\frac{h}{k}\right)\right\}^k = f\left(\frac{h}{k} \times k\right) = f(h) = \{f(1)\}^h, \quad \text{since } h \text{ is}$$

positive integer.

$$\therefore \quad f\left(\frac{h}{k}\right) = \{f(1)\}^{\frac{h}{k}}, \ i.e., f(x) = \{f(1)\}^x.$$

Hence, for all positive values of x, $\{f(1)\}^x = f(x)$.

(iii) Let x be *negative and* $= -y$, so that y is positive.

Now, $f(-y) \times f(y) = f(-y + y) = f(0) = 1$.

$$\therefore \quad f(-y) = \frac{1}{f(y)}.$$

Hence, $f(x) = f(-y) = \dfrac{1}{f(y)} = \dfrac{1}{\{f(1)\}^y}$, since y is positive

$$= \{f(1)\}^{-y} = \{f(1)\}^x.$$

Therefore, whatever x may be, $\{f(1)\}^x = f(x)$.

But $\quad f(1) = 1 + \dfrac{1}{\lfloor 1} + \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 3} + \cdots = e,$

and $\quad f(x) = 1 + \dfrac{x}{\lfloor 1} + \dfrac{x^2}{\lfloor 2} + \dfrac{x^3}{\lfloor 3} + \cdots$

$\therefore$ *for the values of* x

$$e^x = f(x) = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots + \frac{x^r}{\lfloor r} + \cdots$$

104. Expansion of a^x.

To prove that *if a is any positive number*, then

$$a^x = 1 + \frac{x}{1!}(\log_e a) + \frac{x^2}{2!}(\log_e a)^2 + \cdots + \frac{x^r}{r!}(\log_e a)^r + \cdots \text{ to } \infty$$

for all values of x.

If a be any positive number and if k be such that $a = e^k$, so that $k = \log_e a$, then

$$a^x = (e^k)^x = e^{kx} = 1 + \frac{kx}{\lfloor 1} + \frac{k^2 x^2}{\lfloor 2} + \cdots + \frac{k^r x^r}{\lfloor r} + \cdots$$

$$\therefore \quad a^x = 1 + x \log_e a + \frac{x^2}{\lfloor 2}(\log_e a)^2 + \cdots + \frac{x^r}{\lfloor r}(\log_e a)^r + \cdots$$

Otherwise : Let $a^x = e^y$. Then $y = x \log_e a$.

$$\therefore \quad a^x = e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \cdots$$

Note 1. This is known as the **Exponential Series.**

Note 2. If it is required to prove this theorem, the expansion of e^x in power series (Art. 103) should be first established.

105. Illustrative Examples.

Ex. 1. *Find the value of* $\frac{1}{2}\left(e + \frac{1}{e}\right)$ *and* $\frac{1}{2}\left(e - \frac{1}{e}\right)$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

Putting $x = 1$, $e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ $\cdots$ (1)

Putting $x = -1$, $e^{-1} = \frac{1}{e} = 1 - 1 + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \cdots$ $\cdots$ (2)

Adding (1) and (2), and dividing this result by 2, we get

$$\frac{1}{2}\left(e + \frac{1}{e}\right) = 1 + \frac{1}{2!} + \frac{1}{4!} + \cdots$$

Subtracting (2) from (1), and dividing the result by 2, we get

$$\frac{1}{2}\left(e - \frac{1}{e}\right) = 1 + \frac{1}{3!} + \frac{1}{5!} + \cdots$$

Ex. 2. *Show that*

$$1+\frac{1+2}{2!}+\frac{1+2+3}{3!}+\frac{1+2+3+4}{4!}+\cdots\cdots=\tfrac{3}{2}e. \quad [\,C.\,U.\,1935\,]$$

The nth term of the series

$$=\frac{1+2+3+4+\cdots\cdots+n}{n!}$$

$$=\frac{n(n+1)}{2.n!}$$

$$=\tfrac{1}{2}\cdot\frac{(n+1)}{(n-1)!}$$

$$=\tfrac{1}{2}\cdot\frac{(n-1)+2}{(n-1)!}$$

$$=\tfrac{1}{2}\cdot\frac{1}{(n-2)!}+\frac{1}{(n-1)!} \qquad \cdots \quad (1)$$

By inspection,

$$\text{1st term}=\tfrac{1}{2}.0+1,$$

$$\text{2nd term}=\tfrac{1}{2}.1+\frac{1}{1!}.$$

From the relation (1),

$$\text{3rd term}=\tfrac{1}{2}\cdot\frac{1}{1!}+\frac{1}{2!},$$

$$\text{4th term}=\tfrac{1}{2}\cdot\frac{1}{2!}+\frac{1}{3!};$$

and so on.

$\therefore$ By addition, the given series

$$=\tfrac{1}{2}\left\{1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots\right\}+\left\{1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots\right\}$$

$$=\tfrac{1}{2}e+e=\tfrac{3}{2}e.$$

Ex. 3. *Find the value of*

$$1+\frac{1+x}{2!}+\frac{1+x+x^2}{3!}+\frac{1+x+x^2+x^3}{4!}+\cdots\text{ to }\infty$$

Hence, $\qquad t_n = \dfrac{1 + x + x^2 + x^3 + \cdots + x^{n-1}}{n\,!} = \dfrac{1}{n\,!} \cdot \dfrac{1 - x^n}{1 - x}.$

$\therefore \quad t_1 = \dfrac{1}{1\,!} \cdot \dfrac{1 - x}{1 - x}; \quad t_2 = \dfrac{1}{2\,!} \cdot \dfrac{1 - x^2}{1 - x}, \quad t_3 = \dfrac{1}{3\,!} \cdot \dfrac{1 - x^3}{1 - x}; \text{ etc.}$

$\therefore \text{ The series} = \dfrac{1}{1-x}\left[\dfrac{1-x}{1\,!} + \dfrac{1-x^2}{2\,!} + \dfrac{1-x^3}{3\,!} + \cdots \right]$

$\qquad\qquad\qquad = \dfrac{1}{1-x}\left[\left(\dfrac{1}{1\,!} + \dfrac{1}{2\,!} + \dfrac{1}{3\,!} + \cdots \right) - \left(\dfrac{x}{1\,!} + \dfrac{x^2}{2\,!} + \dfrac{x^3}{3\,!} + \cdots \right) \right]$

$\qquad\qquad\qquad = \dfrac{1}{1-x}[(e-1)-(e^x-1)] = \dfrac{e-e^x}{1-x}.$

Ex. 4. *Find the coefficient of x^n in the expansion of* $\dfrac{1-3x+x^2}{e^x}$.

$\dfrac{1-3x+x^2}{e^x} = (1-3x+x^2)e^{-x}$

$\qquad = (1-3x+x^2)\left\{ 1 - x + \dfrac{x^2}{2\,!} - \cdots + (-1)^{n-2}\,\dfrac{x^{n-2}}{(n-2)\,!} \right.$

$\qquad\qquad \left. + (-1)^{n-1}\,\dfrac{x^{n-1}}{(n-1)\,!} + (-1)^n \dfrac{x^n}{n\,!} + \cdots \right\}.$

$\therefore$ The coefficient of x^n $(n > 1)$

$\qquad = (-1)^n \left\{ \dfrac{1}{n\,!} + \dfrac{3}{(n-1)\,!} + \dfrac{1}{(n-2)\,!} \right\}$

$\qquad = (-1)^n \dfrac{1}{n\,!} \left\{ 1 + 3n + n(n-1) \right\} = (-1)^n \cdot \dfrac{(n+1)^2}{n\,!}.$

Ex. 5. *Expand e^{e^x} in ascending powers of x as far as x^4.*

$$[\textit{ C. U. 1938 }]$$

$e^{e^x} = e^{1 + x + \frac{1}{2\,!} x^2 + \cdots} = e \cdot e^{x + \frac{1}{2\,!} x^2 + \cdots}$

$\qquad = e \cdot e^z, \text{ where } z = x + \dfrac{1}{2} x^2 + \cdots$

$\qquad = e\left\{ 1 + z + \dfrac{z^2}{2\,!} + \dfrac{z^3}{3\,!} + \cdots \right\}$

$$= e \left\{ 1 + \left(x + \frac{1}{2\,!}\, x^2 + \cdots \right) + \frac{1}{2\,!} \left(x + \frac{1}{2\,!}\, x^2 + \cdots \right)^2 \right.$$

$$\left. + \frac{1}{3\,!} \left(x + \frac{1}{2\,!}\, x + \cdots \right)^3 + \cdots \right\}$$

$$= e \left\{ 1 + x + x^2 + \tfrac{5}{6} x^3 + \tfrac{5}{8} x^4 + \cdots \right\}$$

on picking out the coefficients of the successive powers of x.

Ex. 6. *Prove that the coefficient of x^r in the expansion of e^{e^x} is*

$$\frac{1}{r\,!} \left\{ \frac{1^r}{1\,!} + \frac{2^r}{2\,!} + \frac{3^r}{3\,!} + \cdots \right\}.$$

Hence, prove that $\dfrac{1^4}{1\,!} + \dfrac{2^4}{2\,!} + \dfrac{3^4}{3\,!} + \cdots$ to $\infty = 15e$.

We have

$$e^{e^x} = 1 + e^x + \frac{e^{2x}}{2\,!} + \frac{e^{3x}}{3\,!} + \frac{e^{4x}}{4\,!} + \cdots$$

$$= 1 + \left(1 + \frac{x^2}{2\,!} + \cdots + \frac{x^r}{r\,!} + \cdots \right) + \frac{1}{2\,!} \left(1 + 2x + \frac{2^2 x^2}{2\,!} + \cdots + \frac{2^r x^r}{r\,!} + \cdots \right)$$

$$+ \frac{1}{3\,!} \left(1 + 3x + \frac{3^2 x^2}{2\,!} + \cdots + \frac{3^r x^r}{r\,!} + \cdots \right) + \cdots$$

$$\therefore \text{ The coefficient of } x^r = \frac{1}{r\,!} \left(\frac{1^r}{1\,!} + \frac{2^r}{2\,!} + \frac{3^r}{3\,!} + \cdots \right) \qquad \cdots \ (1)$$

$$\therefore \text{ The coefficient of } x^4 \text{ is } \frac{1}{4\,!} \left(\frac{1^4}{1\,!} + \frac{2^4}{2\,!} + \frac{3^4}{3\,!} + \cdots \right)$$

and since the coefficient of x^4 is also $\dfrac{5}{8}\,e$, *i.e.*, $\dfrac{15}{4\,!}\,e$ [See *Ex. 5 above*]

therefore, the second part follows.

Examples XII(A)

Show that (*Ex. 1-12*) :

1. $\left\{ 1 + \dfrac{1}{\lfloor 1} + \dfrac{1}{\lfloor 2} + \cdots \right\} \left\{ 1 - \dfrac{1}{\lfloor 1} + \dfrac{1}{\lfloor 2} - \cdots \right\} = 1.$

2. $\left\{ 1 + \dfrac{1}{\lfloor 2} + \dfrac{1}{\lfloor 4} + \cdots \right\}^2 - \left\{ 1 + \dfrac{1}{\lfloor 3} + \dfrac{1}{\lfloor 5} + \cdots \right\}^2 = 1$

3. $x = 1 + \log_e x + \dfrac{(\log_e x)^2}{2\,!} + \dfrac{(\log_e x)^3}{3\,!} + \cdots$

4. $\dfrac{2}{\underline{|3}} + \dfrac{4}{\underline{|5}} + \dfrac{6}{\underline{|7}} + \cdots$ to $\infty = \dfrac{1}{e}$.   [C. U. 1937]

5. $\dfrac{2}{\underline{|1}} + \dfrac{4}{\underline{|3}} + \dfrac{6}{\underline{|5}} + \cdots$ to $\infty = e$.   [C. U. 1936]

6. $\dfrac{1}{2!} + \dfrac{2}{3!} + \dfrac{3}{4!} + \cdots = 1$.

7. $1 + \dfrac{1+2}{\underline{|2}} + \dfrac{1+2+2^2}{\underline{|3}} + \dfrac{1+2+2^2+2^3}{\underline{|4}} + \cdots = e^2 - e$.

     [C. U. 1929]

8. $\left\{ 1 + \dfrac{x^2}{2!} + \dfrac{x^4}{4!} + \cdots \right\}^2 - \left\{ x + \dfrac{x^3}{3!} + \dfrac{x^5}{5!} + \cdots \right\}^2 = 1$.

9. $\left\{ 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \cdots \right\}^2 + \left\{ x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots \right\}^2 = 1$.

10. $\dfrac{1 + \dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \cdots}{1 + \dfrac{1}{3!} + \dfrac{1}{5!} + \dfrac{1}{7!} + \cdots} = \dfrac{e^2 + 1}{e^2 - 1}$.

11. $\dfrac{\dfrac{1}{2!} + \dfrac{1}{4!} + \dfrac{1}{6!} + \cdots}{\dfrac{1}{1!} + \dfrac{1}{3!} + \dfrac{1}{5!} + \cdots} = \dfrac{e-1}{e+1}$.   [C. U. 1934]

12. $\dfrac{1 + \dfrac{2}{2!} + \dfrac{2^2}{3!} + \dfrac{2^3}{4!} + \cdots}{1 + \dfrac{1}{3!} + \dfrac{1}{5!} + \dfrac{1}{7!} + \cdots} = e$.

13. Find the value of

$$(x^2 - y^2) + \dfrac{1}{2!}(x^4 - y^4) + \dfrac{1}{3!}(x^6 - y^6) + \cdots$$

14. Find the value of $\dfrac{1}{\sqrt[5]{e}}$ correct to four places of decimals.   [C. U. 1936].

15. Express in terms of e

(a) $\left(1 + \dfrac{1}{1.2} + \dfrac{1}{1.2.3} + \cdots\right)\left(1 - \dfrac{1}{1.2} + \dfrac{1}{1.2.3} - \cdots\right).$

[C. U. 1938]

(b) $1 + \dfrac{1}{2!} + \dfrac{1.3}{4!} + \dfrac{1.3.5}{6!} + \cdots\cdots$

(c) $\dfrac{1}{1.3} + \dfrac{1}{1.2.3.5} + \dfrac{1}{1.2.3.4.5.7} + \cdots\cdots$

16. Sum to infinity the following series :

(a) $\dfrac{1.2}{1!} + \dfrac{2.3}{2!} + \dfrac{3.4}{3!} + \cdots\cdots$

(b) $1 + \dfrac{3}{1!} + \dfrac{5}{2!} + \dfrac{7}{3!} + \cdots\cdots$

(c) $\dfrac{1^2}{1!} + \dfrac{2^2}{2!} + \dfrac{3^2}{3!} + \cdots\cdots$

(d) $(1+2)\log_e 2 + \dfrac{1+2^2}{2!}(\log_e 2)^2$
$\qquad\qquad + \dfrac{1+2^3}{3!}(\log_e 2)^3 + \cdots$

17. Sum to infinity the series whose nth term is $\dfrac{n^3}{n!}.$

[C. U. 1939]

18. Find the coefficient of x^n in the expansion of

(i) $\dfrac{1 + x + x^2}{e^x}.$ (ii) $\sqrt{\left(\dfrac{x^2 - 4x - 4}{e^x}\right)}.$

19. Show that the coefficient of x^r in the series
$$\dfrac{x+1}{1!} + \dfrac{(x+1)^2}{2!} + \dfrac{(x+1)^3}{3!} + \cdots = \dfrac{e}{r!}.$$

20. Find the coefficient of x^r in the infinite series
$$1 + \dfrac{(a+bx)}{1!} + \dfrac{(a+bx)^2}{2!} + \dfrac{(a+bx)^3}{3!} + \cdots$$

21. If a_n is the coefficient of x^n in the expansion of $\dfrac{e^x}{1-x}$ in ascending powers of x, show that $a_n - a_{n-1} = \dfrac{1}{n\,!}$.

22. Expand $\dfrac{e^{5x} + e^x}{e^{3x}}$ in a series of ascending powers of x.

23. Expand $\dfrac{x}{e^x - 1}$ in ascending powers of x as far as x^4.

24. Expand each of the following in ascending powers of x.

   (i) $\left(1 + \dfrac{x^2}{2\,!} + \dfrac{x^4}{4\,!} + \cdots \right)^2$.

   (ii) $\left(2 + \dfrac{x}{1\,!} + \dfrac{x^2}{2\,!} + \dfrac{x^3}{3\,!} + \cdots \right)^2$.

        [(ii) *The exp.* $= (1 + e^x)^2 = 1 + 2e^x + e^{2x}$]

25. Assuming that when x is *imaginary*

$$e^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \cdots\cdots$$

   (i) Expand $\frac{1}{2}(e^{ix} + e^{-ix})$ in a series of ascending powers of x where $i = \sqrt{-1}$.　　　　　　[*C. U. 1936*]

   (ii) If $a = 1 + \dfrac{x^3}{3\,!} + \dfrac{x^6}{6\,!} + \cdots$

$$b = x + \frac{x^4}{4\,!} + \frac{x^7}{7\,!} + \cdots$$

$$c = \frac{x^2}{2\,!} + \frac{x^5}{5\,!} + \frac{x^8}{8\,!} + \cdots,$$

show that $a^3 + b^3 + c^3 - 3abc = 1$.

   [$a^3 + b^3 + c^3 - 3abc = (a+b+c)(a+\omega b + \omega^2 c)(a + \omega^2 b + \omega c)$]

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13. $e^{x^2} - e^{y^2}$.   **14.** ·8187.   **15.** (a) $e + e^{-1} - 2$.   (b) $\sqrt{e}$.   (c) e^{-1}.

16. (a) $3e$.   (b) $3e$.   (c) $2e$.   (d) 4.   **17.** $5e$.   **18.** (i) $(-1)^n . (n-1)^2/n!$

(ii) $(-1)^{n+1}(n+1)/2^{n-1}.n!$ **20.** $e^a . b^r/r!$

22. $2\left\{1 + \dfrac{2^2 x^2}{2!} + \dfrac{2^4 x^4}{4!} + \cdots \cdots + \dfrac{2^{2n} x^{2n}}{(2n)!} + \cdots \cdots \right\}$,

23. $1 - \frac{3}{2}x + \frac{1}{12}x^2 - \frac{1}{720}x^4 + \cdots \cdots$

24. (i) $\dfrac{1}{2}\left\{2 + \dfrac{2^2 x^2}{2!} + \dfrac{2^4 x^4}{4!} + \cdots \cdots + \dfrac{2^{2n} x^{2n}}{(2n)!} + \cdots \cdots \right\}$.

(ii) $4 + \dfrac{2+2}{1!}x + \dfrac{2^2+2}{2!}x^2 + \dfrac{2^3+2}{3!}x^3 + \cdots + \dfrac{(2^n+2)}{n!}x^n + \cdots$

25. (i) $1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4} - \dfrac{x^6}{6!} + \cdots \cdots + (-1)^r \dfrac{x^{2r}}{(2r)!} + \cdots \cdots$

SEC. B. LOGARITHMIC SERIES

106. Expansion of $\log_e (1+x)$.

By the Exponential Theorem,

$$a^y = 1 + y \log_e a + \frac{y^2}{\underline{|2}} (\log_e a)^2 + \cdots \cdots$$

Putting $1 + x$ for a, we have

$$(1+x)^y = 1 + y \log_e (1+x) + \frac{y^2}{\underline{|2}} \{\log_e (1+x)\}^2 + \cdots \quad (1)$$

By the Binomial Theorem, *if x be numerically less than* 1,

$$(1+x)^y = 1 + yx + \frac{y(y-1)}{\underline{|2}} x^2 + \frac{y(y-1)(y-2)}{\underline{|3}} x^3 + \cdots (2)$$

Hence, from (1) and (2), *if x be numerically less than* 1,

$$1 + y \log_e (1+x) + \frac{y^2}{\underline{|2}} \{\log_e (1+x)\}^2 + \cdots$$

$$= 1 + yx + \frac{y(y-1)}{\underline{|2}} x^2 + \frac{y(y-1)(y-2)}{\underline{|3}} x^3 + \cdots$$

Since this is an identity, coefficients of y on both sides must be equal.

Now, coeff. of y on the left side $= \log_e (1 + x)$ and that on the right side

$$= x + \frac{-1}{\lfloor 2} x^2 + \frac{(-1).(-2)}{\lfloor 3} x^3 + \frac{(-1).(-2).(-3)}{\lfloor 4} x^4 + \cdots$$

$\therefore$ $\log_e (1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \cdots + (-1)^{r-1} \dfrac{x^r}{r} + \cdots$ to ∞ $\quad \cdots$ (3)

if x be numerically less than 1.

Note. Here, the expansion of $\log_e (1+x)$ is established under the assumption that x is numerically less than 1, but *the expansion is also true, if $x=1$.* The proof is beyond the scope of the present work. Thus **the expansion of $\log_e (1+x)$ in ascending powers of x is valid when $-1 < x \leqslant 1$.** The student should note the series for $\log_e 2$, obtained by putting $x=1$ in the series (3) ; thus,

$$\log_e 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots$$

Cor. Since, the expansion of $\log_e (1 + x)$ holds when x is numerically less than 1, we have by putting $-x$ for x in (3),

$$\log_e (1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \cdots - \frac{x^r}{r} - \cdots \text{ to } \infty. \quad \cdots \quad (4)$$

$$(-1 \leqslant x < 1).$$

107. Deductions from Logarithmic Series.

$$\log_e (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\log_e (1 - x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

By subtraction, we get

$$\log_e (1 + x) - \log_e (1 - x) = 2\left\{ x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \right\},$$

or, $\log_e \frac{1+x}{1-x} = 2\left\{x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right\}$ $\qquad$ ... (5)

i.e., $\frac{1}{2}\log_e \frac{1+x}{1-x} = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots \ (-1 < x < 1)$.

Putting $\frac{m-n}{m+n}$ for x, $(m > n)$, so that $\frac{1+x}{1-x} = \frac{m}{n}$,

we get from (5),

$$\log_e \frac{m}{n} = 2\left\{\frac{m-n}{m+n} + \frac{1}{3}\left(\frac{m-n}{m+n}\right)^3 + \frac{1}{5}\left(\frac{m-n}{m+n}\right)^5 + \cdots\right\} \quad (6)$$

Putting $n = 1$, we have

$$\log_e m = 2\left\{\frac{m-1}{m+1} + \frac{1}{3}\left(\frac{m-1}{m+1}\right)^3 + \frac{1}{5}\left(\frac{m-1}{m+1}\right)^5 + \cdots\right\}. \quad (7)$$

Again, putting $m = n + 1$, in (6), we get

$\log_e \frac{n+1}{n} = \log_e (n+1) - \log_e n$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3}\cdot\frac{1}{(2n+1)^3} + \frac{1}{5}\cdot\frac{1}{(2n+1)^5} + \cdots\right\} \cdots \ (8)$$

Again, writing $\frac{1}{n}$ and $-\frac{1}{n}$ for x in $\log_e (1+x)$, we get

$\log_e \frac{n+1}{n}$ and $\log_e \frac{n-1}{n}$ respectively.

$\therefore \ \log_e (n+1) - \log_e n = \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots$ $\quad$ ... (9),

$\log_e n - \log_e (n-1) = \frac{1}{n} + \frac{1}{2n^2} + \frac{1}{3n^3} + \cdots$ $\quad$ ... (10),

where $n > 1$.

Adding (9) and (10), we get

$$\log_e (n+1) - \log_e (n-1) = 2\left\{\frac{1}{n} + \frac{1}{3n^3} + \frac{1}{5n^5} + \cdots\right\}. \quad ... (11)$$

108. Calculation of Logarithms.

The series (7) is suitable for the calculation of the logarithms of small numbers greater than unity.

The series (8) is very convenient for the calculation of the logarithms of one of two consecutive numbers, when the logarithm of the other is known. Thus, putting $n = 1$, in (8),

$$\log_e 2 = 2 \left\{ \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{3}{3^5} + \frac{1}{7} \cdot \frac{1}{3^7} + \cdots \right\}$$

$$= 2 \{ \cdot 33333 + \tfrac{1}{3} \times \cdot 03704 + \tfrac{1}{5} \times \cdot 00432$$
$$+ \tfrac{1}{7} \times \cdot 00046 + \cdots \}$$

$$= \cdot 69315, \text{ correct to 5 places.}$$

Similarly, $\log_e 3 - \log_e 2$

$$= 2 \left\{ \frac{1}{5} + \frac{1}{3} \cdot \frac{1}{3^3} + \frac{1}{5} \cdot \frac{1}{5^5} + \cdots \right\}$$

$$= \cdot 405465 \cdots = \cdot 40547, \text{ correct to 5 places.}$$

$$\therefore \ \log_e 3 = \cdot 69315 + \cdot 40547 = 1 \cdot 09862, \text{ correct to 4 places.}$$

Proceeding in the way, logarithm to the base e of any number can be found to any requisite order of approximation. Having found $\log_e 2$, we have at once $\log_e 4 = \log_e 2^2$ $= 2 \log_e 2$ and we can hence easily find $\log_e 5$; and so on.

Applying this method we can easily find,

$$\log_e 10 = 2 \cdot 30258509 \text{ approximately.}$$

Logarithms can also be calculated by using the series (9), (10) or (11) of the last article.

From Art. 94, we know that, if N be any number

$$\log_e N = \log_{10} N \times \log_e 10.$$

$$\therefore \ \log_{10} N = \log_e N \times \frac{1}{\log_e 10}.$$

With the help of this formula, we can convert the logarithm of any number in the Napierian system (*i.e.*, in which the base is *e*) to the common system (*i.e.*, in which the base is 10).

The *modulus* of the common system

$$= \frac{1}{\log_e 10} = \frac{1}{2 \cdot 30258509} = \cdot 43429448.$$

For sake of brevity, modulus is shortly denoted by μ.

Multiplying the series (8), (9), (10), (11) throughout by μ, we obtain formulæ adapted to the calculation of *common logarithms*.

Thus, from (8), we have

$$\log_{10} (n+1) - \log_{10} n$$
$$= \mu \log_e (n+1) - \mu \log_e n$$
$$= 2\mu \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \cdots \right\},$$

where $\mu = \cdot 43429448$.

Similarly, we can obtain formulæ from (9), (10) and (11).

It should be noted however that the above formulæ are needed for the calculation of the logarithms of *prime* numbers only ; the logarithms of composite numbers can be obtained by adding together the logarithms of the prime factors.

Thus, we see how logarithmic tables can be constructed.

109. Illustrative Examples.

Ex. 1. *Show that* $\log_e 2 = \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$

Putting $x = 1$ in the series for $\log_e (1+x)$, we have

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots$$
$$= (1 - \tfrac{1}{2}) + (\tfrac{1}{3} - \tfrac{1}{4}) + (\tfrac{1}{5} - \tfrac{1}{6}) + \cdots$$
$$= \frac{1}{1.2} + \frac{1}{3.4} + \frac{1}{5.6} + \cdots$$

Ex. 2. *Find the coefficient of x^r in the expansion of $\log_e (1+x+x^2)$.*

$$\log_e (1+x+x^2) = \log_e \frac{1-x^3}{1-x} = \log_e (1-x^3) - \log_e (1-x)$$

$$\log_e (1-x^3) = -x^3 - \tfrac{1}{2}x^6 - \tfrac{1}{3}x^9 - \cdots$$

$$\log_e (1-x) = -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \cdots$$

$$\therefore \quad \log_e (1+x+x^2) = (x+\tfrac{1}{2}x^2+\tfrac{1}{3}x^3+\cdots) - (x^2+\tfrac{1}{2}x^6+\cdots).$$

If r is not a multiple of 3, the coefficient of $x^r = \dfrac{1}{r}$; and if r is a multiple of 3, the coefficient $= \dfrac{1}{r} - \dfrac{3}{r} = -\dfrac{2}{r}$.

Ex. 3. *If a and β be the roots of $x^2 - px + q = 0$, show that*

$$\log_e (1+px+qx^2) = (a+\beta)\,x - \tfrac{1}{2}(a^2+\beta^2)\,x^2 + \tfrac{1}{3}(a^3+\beta^3)\,x^3 - \cdots$$

Since a and β are the roots of $x^2 - px + q = 0$; $\therefore a+\beta = p,\ a\beta = q,$

$\therefore \quad 1+px+qx^2 = 1+(a+\beta)\,x + a\beta x^2 = (1+ax)(1+\beta x).$

$\therefore \quad \log_e (1+px+qx^2)$

$$= \log_e (1+ax) + \log_e (1+\beta x)$$

$$= (ax - \tfrac{1}{2}a^2x^2 + \tfrac{1}{3}a^3x^3 - \cdots) + (\beta x - \tfrac{1}{2}\beta^2x^2 + \tfrac{1}{3}\beta^3x^3 - \cdots)$$

$$= (a+\beta)\,x - \tfrac{1}{2}(a^2+\beta^2)\,x^2 + \tfrac{1}{3}(a^3+\beta^3)\,x^3 - \cdots$$

Ex. 4. *If $y = x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots$, show that*

$$x = y + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \cdots$$

[*C. U. 1931*]

Here $y = \log_e (1+x)$. $\therefore 1+x = e^y$. $\therefore x = e^y - 1$.

$$\therefore \quad x = \left(1 + y + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \cdots\right) - 1 = y + \frac{y^2}{2\,!} + \frac{y^3}{3\,!} + \cdots$$

Examples XII(B)

Prove that (*Ex. 1-8*) :

1. $\dfrac{1}{n+1} + \dfrac{1}{2\,(n+1)^2} + \dfrac{1}{3\,(n+1)^3} + \cdots = \dfrac{1}{n} - \dfrac{1}{2n^2} + \dfrac{1}{3n^3} - \cdots$

[*C. U. 1945*]

2. $\log_e (1-x) = -2\{\tfrac{1}{2}x^2 + \tfrac{1}{4}x^4 + \tfrac{1}{6}x^6 + \cdots\}.$

3. $\left(\dfrac{1}{3} - \dfrac{1}{2}\right) + \dfrac{1}{2}\left(\dfrac{1}{3^2} + \dfrac{1}{2^2}\right) + \dfrac{1}{3}\left(\dfrac{1}{3^3} - \dfrac{1}{2^3}\right) + \cdots = 0.$

4. $\dfrac{x - y}{x} + \dfrac{1}{2}\left(\dfrac{x - y}{x}\right)^2 + \dfrac{1}{3}\left(\dfrac{x - y}{x}\right)^3 + \cdots = \log_e \dfrac{x}{y}.$

5. $\dfrac{1}{1.2} + \dfrac{1}{2.2^2} + \dfrac{1}{3.2^3} + \cdots = \log_e 2.$

6. $\dfrac{1}{2.3} + \dfrac{1}{4.5} + \dfrac{1}{6.7} + \cdots = 1 - \log_e 2.$ [C. U. 1940]

7. $\left(\dfrac{1}{5} + \dfrac{1}{7}\right) + \dfrac{1}{3}\left(\dfrac{1}{5^3} + \dfrac{1}{7^3}\right) + \dfrac{1}{5}\left(\dfrac{1}{5^5} + \dfrac{1}{7^5}\right) + \cdots = \log_e \sqrt{2}.$

8. $\log_e x = \dfrac{x - 1}{x + 1} + \dfrac{1}{2}\dfrac{x^2 - 1}{(x + 1)^2} + \dfrac{1}{3}\dfrac{x^3 - 1}{(x + 1)^3} + \cdots, \; x > 0.$

9. Expand $\log_e (1 - 3x + 2x^2)$ and $\log_e \dfrac{1 - 3x + 2x^2}{1 - 4x + 3x^2}$ in ascending powers of x, and find the coefficient of x^n in each case.

10. Expand $\log_e \dfrac{1 + x + x^2}{1 - x + x^2}$ in ascending powers of x.

11. Expand $\log_e (1 + x + x^2 + x^3 + \cdots\cdots \text{ to } \infty)$ in ascending powers of x and find the coefficient of x^n, if $x^2 < 1$.

12. Find the relation between P and Q, if

$$P = \dfrac{x}{1 + x^2} + \dfrac{1}{3}\left(\dfrac{x}{1 + x^2}\right)^3 + \dfrac{1}{5}\left(\dfrac{x}{1 + x^2}\right)^5 + \cdots$$

and $Q = x - \dfrac{2}{3} x^3 + \dfrac{1}{5} x^5 + \dfrac{1}{7} x^7 - \dfrac{2}{9} x^9 + \cdots$

13. Expand $\log_e (1 + x + x^2 + x^3)$ in ascending powers of x and find the coefficients of x^{2n+1} and x^{2n}.

14. If $\log_e \dfrac{1}{1 - x - x^2 + x^3}$ be expanded in a series of ascending powers of x, show that the coefficient of x^n is $\dfrac{1}{n}$ or $\dfrac{2}{n}$ according as n is odd or even.

Sum to infinity the following series $(x^2 < 1)$ *(Ex. 15-18)* :

15. $\dfrac{1}{2}\, x^2 + \dfrac{1}{4}\, x^4 + \dfrac{1}{6}\, x^6 + \cdots$

16. $\dfrac{1}{2}\, x^2 + \dfrac{2}{3}\, x^3 + \dfrac{3}{4}\, x^4 + \cdots$ **17.** $\dfrac{x}{1.2} + \dfrac{x^2}{2.3} + \dfrac{x^3}{3.4} + \cdots$

18. $1 + \dfrac{1}{3.2^2} + \dfrac{1}{5.2^4} + \dfrac{1}{7.2^6} + \cdots$ [*C. U. 1943*]

19. Prove that
$$\log_e (1+x)^{1+x}\,(1-x)^{1-x} = 2\left(\dfrac{x^2}{1.2} + \dfrac{x^4}{3.4} + \dfrac{x^6}{5.6} + \cdots\right).$$

20. If α and β are the roots of the equation $ax^2 + bx + c = 0$, show that $\log_e(a - bx + cx^2)$
$$= \log_e a + (\alpha + \beta)\, x - \tfrac{1}{2}(\alpha^2 + \beta^2)\, x^2 + \tfrac{1}{3}(\alpha^3 + \beta^3)\, x^3 - \cdots$$

21. Prove that under certain conditions (to be stated)
$$\dfrac{1}{x^2} + \dfrac{1}{2x^4} + \dfrac{1}{3x^6} + \cdots = \dfrac{2}{y} + \dfrac{2}{3y^3} + \dfrac{2}{5y^5} + \cdots,$$
where $y = 2x^2 - 1$.

22. If a, b, c are H. P. and $a > b > c$, then
$$\left(\dfrac{c}{b} - \dfrac{b}{a}\right) + \dfrac{1}{2}\left(\dfrac{c^2}{b^2} - \dfrac{b^2}{a^2}\right) + \dfrac{1}{3}\left(\dfrac{c^3}{b^3} - \dfrac{b^3}{a^3}\right) + \cdots = \log_e \dfrac{b}{c}\,.$$

23. If $y = x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \tfrac{1}{4}x^4 + \cdots$, show that
$$y = y - \dfrac{1}{2\,!}\, y^2 + \dfrac{1}{3\,!}\, y^3 - \dfrac{1}{4\,!}\, y^4 + \cdots$$

24. If $0 < x < 1$, and if $y = \dfrac{x}{x+1}$, express the sum to infinity of $x - \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 - \cdots$ as a power series in y.

25. Show that $\log_e (1 + 2x + 3x^2 + 4x^3 + \cdots)$, $x^2 < 1$,

$$= 2 \left(x + \frac{1}{2} x^2 + \frac{1}{3} x^3 + \cdots + \frac{1}{n} x^n + \cdots \right).$$

26. If $x^2 < 1$, expand $\{\log_e (1 + x)\}^2$ in ascending powers of x.

[*Equate the cofficients of y^2 on both sides of (3), Art. 106*]

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9. (i) $-\left\{ \dfrac{2+1}{1}x + \dfrac{2^2+1}{2}x^2 + \cdots\cdots + \dfrac{2^n+1}{n} x^n + \cdots \right\}.$

(ii) $\dfrac{3-2}{1}x + \dfrac{3^2-2^2}{2}x^2 + \cdots\cdots + \dfrac{3^n-2^n}{n} x^n + \cdots\cdots$

10. $2 \left[x - \frac{2}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 - \frac{2}{9}x^9 + \cdots \right].$

11. $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots\cdots + \dfrac{1}{n} x^n + \cdots$; coeff. of $x^n = \dfrac{1}{n}.$ **12.** $P = Q.$

13. $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{2}{4}x^4 + \cdots\cdots$; coeff. of $x^{2n+1} = \dfrac{1}{2n+1}$;

coeff. of. $x^{2n} = -\dfrac{1}{2n} + (-1)^{n-1} \dfrac{1}{n}.$

15. $-\frac{1}{2} \log_e (1-x^2).$ **16.** $\dfrac{x}{1-x} + \log_e (1-x).$

17. $1 + \dfrac{1-x}{x} \log_e (1-x).$ **18.** $\log_e 3.$ **25.** $y + \frac{1}{2}y + \frac{1}{3}y^3 + \cdots\cdots$

26. $2\{\frac{1}{2}x^2 - \frac{1}{3}(1 + \frac{1}{2})x^3 + \frac{1}{4}(1 + \frac{1}{2} + \frac{1}{3})x^4 - \cdots\cdots\}.$

INTEREST AND ANNUITIES

110. Introduction.

When money is lent, the borrower usually pays to the lender a certain sum in consideration for the use of it. The sum of money lent is called the *Principal* ; the sum of money paid for the use of it is called the *Interest* and the sum of the principal and the interest due at any time is called the *Amount* at that time.

The *rate* of interest is the sum paid for the use of a certain sum for a certain time ; thus, if 4as. be paid for the use of 1 rupee for 1 year, the rate of interest is said to be 4as. per rupee per year.

The *rate per cent. per annum* is the sum paid as interest for the use of Rs. 100 for 1 year ; thus, if Rs. 3 be paid as interest for Rs. 100 for 1 year, the rate of interest is said to be 3 per cent. per annum.

When the principal alone produces interest, it is called *Simple Interest* ; but when the interest, as soon as it becomes due, is added to the principal and the whole then produces interest, is called *Compound Interest*.

The *Present Worth* or *Present Value* of and amount due at the end of a given period of time is that sum which together with its interest for the period is equal to the given amount.

Discount is the abatement made for the payment of money before it is due. Hence, discount is the difference between the amount due at the end of the given time and the present worth.

111. Simple Interest Formula.

Let P rupees be the principal, n the number of years for which the principal is lent, r rupees the interest on Re. 1 for 1 year. Let I rupees be the interest on P rupees for n years and M rupees be the amount of P rupees in n years.

Interest   on   1 rupee   for   1 year $= r$ rupees.

$\therefore$ "   " P rupees   "   "   " $= Pr$ "

$\therefore$ "   " P "   " n years $= Pnr$ "

$\therefore$ $\qquad$ $I = Pnr$ $\cdots$ $\cdots$ (1)

$\therefore$ $M = P + I = P + Pnr = P(1 + nr).$ $\cdots$ (2)

Thus, we see that *in the case of simple interest, the amount increases in A. P.*

Cor. If c denotes the *rate per cent.*, then $\dfrac{c}{100} = r$; hence, the above formulæ become

$$I = \frac{Pnc}{100} \; ; \; M = P\left(1 + \frac{nc}{100}\right).$$

Note. The above formulæ would remain perfectly true if any other unit of money, say pound or shilling is substituted for rupee.

112. Compound Interest Formula *(conversion per year)*

Let P rupees be the principal, n the number of years for which the principal is lent, r rupees the interest on Re. 1 for 1 year. I rupees the interest on P rupees for n years, and M rupees the amount at the end of n years.

Let the interest be converted into principal at the end of every year.

Interest   on   1 rupee   for   1 year $= r$ rupees

$\therefore$ "   " P rupees   "   "   " $= Pr$ "

$\therefore$ Amount   of P "   in   1 year $= P + Pr$

$\qquad\qquad\qquad\qquad$ or, $P(1 + r)$ rupees.

∴   Amount of P rupees in 2 years
$$= \text{amount of } P(1+r) \text{ in 1 year}$$
$$= P(1+r) . (1+r) = P(1+r)^2.$$

Amount of P rupees in 3 years
$$= \text{amount of } P(1+r)^2 \text{ in 1 year}$$
$$= P(1+r)^2 . (1+r) = P(1+r)^3.$$

∴   Amount of P rupees in n years $= P(1+r)^n$.

∴ $$M = P(1+r)^n, \qquad \ldots \qquad \ldots \quad (3)$$
$$I = M - P = P\{(1+r)^n - 1\} \qquad \ldots \quad (4)$$

Putting R for $(1+r)$, so that R *becomes the amount of 1 rupee for 1 year*, the above formulæ are sometimes written in the forms ;

$$M = PR^n, \qquad \ldots \qquad \ldots \quad (3')$$
$$I = P(R^n - 1). \qquad \ldots \qquad \ldots \quad (4')$$

Thus, *in the case of compound interest, the amount increases in geometrical progression.*

Cor.   If c is the rate per cent, the formulæ (3) and (4) become

$$M = P\left(1 + \frac{c}{100}\right)^n ; \quad I = P\left\{\left(1 + \frac{c}{100}\right)^n - 1\right\}.$$

Note. To facilitate arithmetical work, it is customary to use logarithms in the calculation of compound interest. Thus, taking logarithms of both sides of (3), we have
$$\log M = \log P + n \log(1+r),$$
an equation from which any one of the four quantities M, P, n, r can be readily obtained when the remaining three are given.

113.   Compound Interest Formula (*conversion per part of a year*).

In the above formula we have assumed that the interest accruing from the use of money has been converted into

the principal at the end of every year ; but if the conversion of interest takes place at the end of every qth part of a year, *i.e.*, if the interest is paid q times in a year, we shall have with the notation of the previous article,

$$1 + \frac{r}{q} = \text{amount of Re. 1 at the end of}$$

qth part of the year

$$\therefore \quad P\left(1 + \frac{r}{q}\right) = \quad \cdots \quad \text{Rs. } P \quad \cdots \quad \cdots \quad \cdots$$

$$\therefore \quad P\left(1 + \frac{r}{q}\right)^q = \quad \cdots \quad \text{Rs. } P \quad \cdots \quad \cdots \quad 1 \text{ year}$$

$$\therefore \quad P\left(1 + \frac{r}{q}\right)^{an} = \quad \cdots \quad \text{Rs. } P \quad \cdots \quad \cdots \quad n \text{ years}$$

$\therefore$ The total amount in n years $= P\left(1 + \frac{r}{q}\right)^{an}$;

$$\therefore \quad M = P\left(1 + \frac{r}{q}\right)^{qn}$$

$$I = M - P = P\left\{\left(1 + \frac{r}{q}\right)^{qn} - 1\right\}.$$

Note 1. Thus, when the interest is paid *half-yearly, quarterly, monthly,* we can have for the amount in n years,

$$M = P\ (1 + \tfrac{1}{2}r)^{2n}, \ M = P\ (1 + \tfrac{1}{4}r)^{4n}, \ M = P\ (1 + \tfrac{1}{12}r)^{12n}$$

respectively.

Note 2. When q is made infinitely large in the above formula, we get

$$\underset{p \to \infty}{Lt}\ \left(1 + \frac{r}{q}\right)^{qn} = e^{nr}. \quad [\ Art.\ 108,\ Cor.\ 2\ ]$$

$$\therefore \quad M = Pe^{nr}.$$

When q becomes indefinitely great, the intervals between the payments become infinitely small ; so in this case, the interest becomes due every moment and is converted into the principal continuously.

114. Present Value and Discount.

To find the present value and discount of a given sum due in a given time.

Let V be the present value of the sum P due at the end of n years, r the interest on Re. 1 for 1 year.

(i) *If simple interest is allowed* :

Since, the sum V, put to simple interest, amounts to P in n years,

$\therefore$ $P = V(1 + nr)$.   [*Art. 111*]

$\therefore$ The present value $V = \dfrac{P}{1+nr}$.

Now $D = P - V = P - \dfrac{P}{1+nr} = \dfrac{Pnr}{1+nr}$.

$\therefore$ The discount $D = \dfrac{Pnr}{1+nr}$.

Note. In practice the bankers and merchants invariably deduct the *interest on the whole debt* from the present time till it becomes due instead of the interest of the present worth, which is the true discount. Thus,

$$\text{True discount} = \frac{Pnr}{1+nr} \text{ ; and } \text{Banker's discount} = Pnr.$$

(ii) *If compound interest is allowed* :

Then, $P = V(1+r)^n = VR^n$.   [*Art. 112*]

$\therefore$ $V = \dfrac{P}{(1+r)^n} = \dfrac{P}{R^n}$;

since, $D = P - V = P - \dfrac{P}{(1+r)^n} = P\left\{1 - \dfrac{1}{(1+r)^n}\right\}$.

$\therefore$ $D = \dfrac{P\{(1+r)^n - 1\}}{(1+r)^n} = \dfrac{P(R^n - 1)}{R^n}$.

115. Illustrative Examples.

Ex. 1. *In how many years will a sum of money double itself at 5 per cent. per annum compound interest ?*

[Given $\log 2 = \cdot 3010300$, $\log 3 = \cdot 4771213$, $\log 7 = \cdot 8450980$].

Let P be the sum of money and n be the required number of years.

Then, $P(1 + \tfrac{5}{100})^n = 2P$. [*Art. 112, Cor.*]

$\therefore \left(\tfrac{21}{20}\right)^n = 2$.

$\therefore n(\log 21 - \log 20) = \log 2$,

or, $n(\log 3 + \log 7 - \log 10 - \log 2) = \log 2$.

Now, substituting the value of $\log 2$, $\log 3$, $\log 7$ and noting that $\log 10 = 1$,

we get $\qquad n = \dfrac{\cdot 3010300}{\cdot 0211893} = 14 \cdot 2$ approximately.

$\therefore$ The reqd. no. of years $= 14 \cdot 2$ nearly.

Ex. 2. *Find the discount of Rs. 4630. 8as. due 3 years hence at 5 per cent. per annum compound interest.*

By the formula of Art. 114(ii),

$\text{Discount} = \dfrac{4630\frac{1}{2}\{(1 + \frac{1}{20})^3 - 1\}}{(1 + \frac{1}{20})^3}$, since, $r = \tfrac{5}{100} = \tfrac{1}{20}$

$\qquad = \dfrac{9261}{2} \cdot \dfrac{\{21^3 - 20^3\}}{21^3} = \dfrac{1}{2} \cdot (9261 - 8000)$

$\qquad = 630\frac{1}{2} = \text{Rs. 630. 8as.}$

Examples XIII(A)

[*The following logarithms whenever required should be used.*

$\log \quad 2 = \cdot 3010300$	$\log 130 = 2 \cdot 0123872$
$\log \quad 3 = \cdot 4771213$	$\log 104 = 2 \cdot 0170333$
$\log \quad 7 = \cdot 8450980$	$\log 105 = 2 \cdot 0211893$
$\log 102 = 2 \cdot 0086002$	$\log 106 = 2 \cdot 0253059$]

1. In what time will a sum of money treble itself at 5 per cent. per annum compound interest ?

2. Find in how many years Rs. 100 will become Rs. 1000 at 4 p.c. compound interest. Give the answer correct to two places of decimals. [*C. U. 1934*]

3. A man invests £30 a year in a Savings Bank which pays $2\frac{1}{2}$ per cent. per annum on all deposits. What will be the total amount at the end of 20 years ? [*C. U. 1937*]
[Given log 1·025 = ·0107239 ; log 1·63862 = ·2144780]

4. Show that a sum of money will increase more than 100 times in 100 years at 5 per cent. compound interest.

5. What sum of money put out at 4 per cent. per annum at compound interest for 18 years will amount to Rs. 10000 ?

[Given log 493629 = 5·6934006]

6. A sum of £100 is put out at compound interest for 2 years ; what is the difference in the amount according as the interest is paid yearly or half-yearly at 4 per cent ?
[Given log 1·0816 = ·0340666 ; log 1·08243 = ·0344008]

7. Supposing the interest to be paid *half-yearly*, what would be the amount of Rs. 100000 put out for 10 years at 10 per cent. per annum compound interest ?

[Given log 26532975 = 7·4237860]

8. If a sum of money at a given rate of compound interest accumulates to m times its original value in x years, and to n times its original value in y years, prove that $y = x \log_m n$.

9. If A, B, C be the amounts at compound interest of a sum of money in a, b, c years respectively, prove that

$$(b - c) \log A + (c - a) \log B + (a - b) \log C = 0.$$

10. How long will it take for one penny to amount to £1000 at 5% compound interest convertible annually ?

11. A person borrows a sum of money, and pays off at the end of each year as much of the principal as he pays

interest for that year ; find how much he owes at the end of n years.

12. Show that the difference between banker's discount and true discount, simple interest being reckoned, is

$$Pn^2r^2 \{1 - nr + n^2r^2 - n^3r^3 + \cdots \text{ to } \infty\}, nr < 1.$$

13. Show that the discount is half the harmonic mean between the sum due and the interest on it, whether simple or compound interest is reckoned.

14. Find the present value and discount of a debt of Rs. 1800 due 5 years hence at 4 per cent. compound interest.

[Given log 18 = 1·2552725 ; log 1479·47 = 3·1701060]

15. Find what sum will amount to £500 in 20 years at 5 per cent., the interest being supposed to be payable every instant.

[Given $e^{-1} = ·3678$]

ANSWERS

1. 22·5 years.   **2.** 58·71 years.   **3.** £785. 18s.

5. Rs. 4936. 4as. 7·68p.   **6.** 1s. 8d. nearly.

7. Rs. 265329. 12as.   **10.** 254 years.   **11.** $P(1-r)^n$.

14. V = Rs. 1479. 7as. 6p. nearly ; D = Rs. 320. 3as. 6p. nearly.

15. £183. 18s.

SEC. B. ANNUITIES

117. Definition.

An *annuity* is a fixed sum of money (Interest, Rent or Pension) usually payable annually under certain stated conditions. In some cases, it is also payable at more, frequent intervals, say half-yearly, quarterly etc.

If any annuity is left unpaid for a number of years, it is said to be unpaid or *forborne* for that number of years

and the annual sums together with the interest on each for the *amount* of the annuity.

If an annuity is payable for a fixed number of years, it is said to be an *annuity certain*. If an annuity is to continue forever, it is said to be a *perpetual annuity* or *perpetuity*.

If an annuity does not commence until after the lapse of a certain time, it is called a *deferred annuity*. When an annuity is deferred for n years, it is said to commence after n years and the first payment is made after $(n + 1)$ years.

The *present value* of annuity which is to continue for a given number of years is that sum which together with its interest for that time is equal to the amount of the annuity.

An estate which is held for a fixed number of years is called a *lease-hold estate* and the rent in this case is a terminable annuity, whereas a free-hold estate yields a perpetual annuity called the *rent*. The *value of* a *free-hold estate* is the present value of this perpetuity *viz.*, the rent.

118. Amount of Annuity left unpaid.

(i) *To find the amount of an annuity left unpaid for a given number of years, allowing simple interest.*

Let A be the annuity, n the number of years, r the interest on Re. 1 for 1 year, M the amount.

The first payment due is A at the end of the 1st year, and since it remains unpaid for $(n - 1)$ years, the amount of this sum in $(n - 1)$ years at the above rate of simple interest is $A\{1 + (n - 1)r\}$.

The 2nd payment due is A at the end of the 2nd year, and since it remains unpaid for $(n - 2)$ years, the amount becomes $A\{1 + (n - 2)r\}$.

Similarly, the amount of 3rd payment $= A\{1 + (n - 3)r\}$; and so on.

$\therefore$ M, *i.e.*, the whole amount due at the end of n years

$$= A\{1 + (n-1)\,r\} + A\{1 + (n-2)\,r\} + \cdots \text{ to } n \text{ terms}$$
$$= nA + Ar\{(n-1) + (n-2) + \cdots + 2 + 1\}$$
$$= nA + Ar\{1 + 2 + \cdots + (n-1)\}$$
$$= nA + \tfrac{1}{2}n\,(n-1)\,rA.$$

(ii) *To find the amount of an annuity left unpaid for a given number of years, allowing compound interest.*

With the same notation as before, we have

the amount of the 1st payment $= A\,(1+r)^{n-1}$

$\cdots$ $\qquad$ $\cdots$ $\qquad$ 2nd $\quad \cdots$ $\quad = A\,(1+r)^{n-2}$

$\cdots$ $\qquad$ $\cdots$ $\qquad$ 3rd $\quad \cdots$ $\quad = A\,(1+r)^{n-3}$

and so on.

$\therefore$ $M = A\{(1+r)^{n-1} + (1+r)^{n-2} + \cdots + (1+r) + 1\}$

$$= A\{1 + (1+r) + (1+r)^2 + \cdots + (1+r)^{n-1}\}.$$

$\therefore$ $M = A \cdot \dfrac{(1+r)^n - 1}{r} = A\,\dfrac{R^n - 1}{R - 1},$

where R is the amount of 1 rupee for 1 year.

118.   Present value of an annuity.

To find the present value of an annuity to continue for a given number of years, allowing compound interest.

Let A be the annuity, R the amount of Re. 1 in 1 year, n the number of years and V the required present value.

The present value of A due in 1 year is AR^{-1} [*Art. 114*]

$\cdots$ $\qquad$ $\cdots$ $\qquad$ $\cdots$ $\qquad$ $\cdots$ $\qquad$ 2 years is AR^{-2}

$\cdots$ $\qquad$ $\cdots$ $\qquad$ $\cdots$ $\qquad$ $\cdots$ $\qquad$ 3 $\qquad$ is AR^{-3}

and so on.

$$\therefore \quad V = AR^{-1} + AR^{-2} + AR^{-3} + \cdots \cdots \text{ to } n \text{ terms}$$

$$= AR^{-1} \left(1 + R^{-1} + R^{-2} + \cdots \cdots \text{ to } n \text{ terms}\right)$$

$$= AR^{-1} \cdot \frac{1 - R^{-n}}{1 - R^{-1}} = \frac{A}{R-1} \left(1 - \frac{1}{R^n}\right).$$

$$\therefore \quad V = \frac{A}{R-1}\left(1 - \frac{1}{R^n}\right) = \frac{A}{r}\left\{1 - \frac{1}{(1+r)^n}\right\}.$$

Otherwise : The amount of V in n years will be VR^n.

[See *Art. 112*]

$\therefore$ We must have

$$VR^n = A \cdot \frac{R^n - 1}{R - 1}. \quad [\text{ See } Art.\ 117]$$

$$\therefore \quad V = \frac{A(R^n - 1)}{R^n(R-1)} = \frac{A}{R-1}\left(1 - \frac{1}{R^n}\right).$$

Cor. By making n infinitely large in the above formula, *the present value of a perpetual annuity* is found to be

$$V = \frac{A}{R-1} = \frac{A}{r},$$

since when n becomes infinitely large, $1/R^n$ tends to zero.

Note 1. If mA is the present value of an annuity A, the annuity is said to be *worth m years' purchase*.

Note 2. In calculating the present value of annuities, it is always customary to allow compound interest.

119. Present value of a Deferred Annuity.

To find the present value of a deferred annuity commence at the end of m years and to continue for n years, allowing compound interest.

Let A be the annuity, R the amount of Re. 1 in 1 year, V the present value.

Since, the first payment is made at the end of $(m + 1)$ years [See *Art. 116*], the second payment at the end of $(m + 2)$ years, and so on, the present values of the first, second,...... payments are respectively

$$AR^{-(m+1)}, AR^{-(m+2)},......AR^{-(m+n)}.$$

$\therefore \quad V =$ sum of the present values of the different payments

$$= AR^{-(m+1)} + AR^{-(m+2)} + AR^{-(m+3)} + \cdots \text{ to } n \text{ terms}$$

$$= AR^{-(m+1)} \{1 + R^{-1} + R^{-2} + \cdots \text{ to } n \text{ terms}\}$$

$$AR^{-(m+1)} \cdot \frac{1 - R^{-n}}{1 - R^{-1}} = \frac{A}{R^{m+n}} \frac{R^n - 1}{R - 1}.$$

$$\therefore \quad v = \frac{A}{R-1} \left\{ \left(1 - \frac{1}{R^{m+n}}\right) - \left(1 - \frac{1}{R^m}\right) \right\}$$

= *the present value of annuity for* $(m + n)$ *years*
— the present value of m *years.*

Cor. By making n infinitely large, *the present value of a deferred perpetuity to commence after* m *years* is given by

$$V = \frac{AR^{-m}}{R-1}$$

120. Illustrative Examples.

Ex. 1. *Find the present value of an annuity of Rs.* 300 *per annum for* 5 *years at* 4 *per cent.*

[*Given log* $1 \cdot 04 = \cdot 0170333$, *log* $821 \cdot 923 = 2 \cdot 9148335$]

Substituting $A = 300$, $r = \frac{4}{100} = \cdot 04$, $n = 5$ in the formula

$$V = \frac{A}{r} \left\{ 1 - \frac{1}{(1+r)^n} \right\}, \quad [\text{ See } \textit{Art. 118}], \text{ we have}$$

$$V = 300 \cdot \frac{100}{4} \left\{ 1 - \frac{1}{(1 + \cdot 04)^5} \right\} = \frac{30000}{4} \{1 - (1 \cdot 04)^{-5}\}.$$

Let us find the value of $(1\cdot04)^{-5}$.

$$x=(1\cdot04)^{-5}.$$

$\therefore$ $\log x= -5 \log 1\cdot04 = -5 \times \cdot0170333 = -\cdot0851665$

$\qquad = \bar{1}\cdot9148335 = \log \cdot821923 \; ;$

$\therefore$ $x = \cdot821923.$

Hence, V (*i.e.*, the reqd. present value) $= \dfrac{30000}{4} (1 - \cdot821923)$

$\qquad = \dfrac{30000}{4} \times \cdot178077 = 1335\cdot58$ nearly

$\qquad = Rs.$ 1335. 9*as.* $2\cdot9$ *p.* nearly.

Ex. 2. *The annual rent of a free-hold estate is Rs. 1000 ; what is its value, allowing 4 per cent. per annum compound interest ?*

Since, the value of a free-hold estate is the present value of the perpetuity, *viz.*, the rent [*Art. 116*], therefore, using the formula $V = \dfrac{A}{r}$ [*Art. 118, Cor.*] and noting that here $A = 1000$ and $r = \frac{4}{100} = \frac{1}{25}$, we get the value of the estate $= \dfrac{1000}{\frac{1}{25}} = Rs.$ 25000.

Examples XIII(B)

[*The logarithms given at the beginning of Ex. XIII(A) should be used whenever required.*]

1. Find the amount of an annuity of Rs. 100 left unpaid for 10 years, allowing 5 per cent. per annum compound interest.

[Given $\log 1\cdot6289 = \cdot211893$]

2. (i) Find the present value of an annuity of Rs. 4000, to continue for 25 years, allowing 5 per cent. per annum compound interest.

[Given $\log 2\cdot95362 = \cdot4702675$]

(ii) What sum should be paid for an annuity of £100 a year to be paid for 40 years, money being supposed to be worth 4 per cent. per annum ? [*C. U. 1935*]

[Given $\log 26 = 1\cdot41497$, $\log 20835 = 4\cdot31880$]

(iii) A loan of Rs. 1000 is to be paid off in 10 equal annual instalments. Find the yearly payment, allowing compound interest at 4 per cent. per annum.

[Given log 1·480238 = ·170333]

3. A debt of Re. 1 accumulating at compound interest is discharged in n years by annual payment of Re. $1/m$.

Prove that $n = -\dfrac{\log (1 - mr)}{\log (1 + r)}$,

where r is the interest on Re. 1 for 1 year.

4. A company borrows Rs. 10000 on condition to repay it with compound interest at 5 per cent. by annual instalments of Rs. 1000 each. In how many years will the debt be paid off ?

5. A person puts his whole fortune Rs. 20000 out at compound interest at 5 per cent. per annum and requires for his annual expenses Rs. 1800. If he begins to spend from the end of the first year and goes on spending at this rate, show that he will be ruined before the end of the 17th year.

6. The annual subscription of a Mathematical Society payable at the beginning of each year is Rs. 10, but a member can compound for all his future subscriptions by paying at a time Rs. 200. If he is satisfied with 3 per cent. interest for his money, find how many years he should live in order that it may be worth his while to compound.

[log 43 = 1·63347]

7. A man aged 60 years receiving a pension of Rs. 1200 per year wishes to commute it for a present payment, interest being reckoned at 4 per cent. per annum. How much does he receive in his expectation of life is 10 years ?

[log 1480238 = 6·170333]

8. A person bought a free-hold estate at Rs. 40000. What should be the net annual income from the estate if money makes 5 per cent. per annum ?

9. If 20 years' purchase must be paid for an annuity to continue a certain number of years and 26 years' purchase for any annuity to continue twice as long, find the rate of interest.

10. If x, y, z years' purchase must be paid for an annuity to continue n, $2n$, $3n$ years respectively, show that

$$x^2 + y^2 = xy + xz.$$

ANSWERS

1. Rs. 1257·8 nearly.

   (ii) £1979. 2s. 6d. nearly.

4. 14·2 years.

8. 2000.

2. (i) Rs. 56371 nearly.

   (iii) Rs. 123. 4as. 8p. nearly.

6. 30 years.   **7.** Rs. 9733 nearly.

9. $3\frac{1}{2}$ p. c.

APPENDIX I

SEC. A. ARITHMETICAL PROGRESSION

1. Series.

A succession of quantities each of which is formed according to some fixed law is called *series*. The successive terms are called the *terms* of the series.

Thus, (i) 1, 3, 5, 7,......,   (ii) 2, 4, 8, 16,......
are examples of series. In (i) each term is formed by adding 2 to the preceding term and in (ii) each term is formed by multiplying the preceding term by 2.

It is clear when the law of formation of a series is known, we can write down any number of terms of the series.

2. Arithmetical Progressions.

A series of quantities is said to be in *Arithmetical Progression* when the algebraic difference between any term and the preceding one is the same throughout the series,

Thus, a, b, c, d etc. are said to be in Arithmetical Progression if $b - a = c - b = d - c =$ etc.

The algebraic difference between each term and the preceding is called the *Common difference*.

Thus, the series (i) 1, 3, 5, 7,...   (ii) 3, -1, -5, -9,...
are in Arithmetical Progression, the common difference in (i) is 2 and that in (ii) is -4.

Note 1. For the sake of convenience, the words 'Arithmetical Progression' and 'common difference' are shortly abbreviated as A. P. and c. d. respectively.

Note 2. From the definition of an A. P., it is clear that each term is formed from the preceding by adding to it a constant quantity *viz.*, the common difference. So an A. P. may be called a *constant difference series.*

3.   The nth term.

Let a be the first term and d the common difference of an A.P.   Then by definition

$$\text{2nd term} = r + d\ = a + (2 - 1)\ d$$
$$\text{3rd term} = a + 2d = a + (3 - 1)\ d$$
$$\text{4th term} = a + 3d = a + (4 - 1)\ d$$

$$\therefore \quad \text{nth term } (t_n) \quad = a + (n - 1)d. \qquad \cdots \quad (1)$$

Thus, *we can write down any term of an A. P. when the first term and the common difference are given.*

Thus, if the first term *i.e.*, $a = 5$ and the common difference *i.e.*, $d = 4$, then the 20th term of the series.

$$\text{i.e., } t_{20} = 5 + (20 - 1) \times 4 = 81.$$

If a be the first term, d the c.d., n the number of terms, and l the last term (*i.e.*, the nth term), then

$$\text{I} = a + (n - 1)d. \qquad \cdots \qquad \cdots \quad (2)$$

Cor. *When any two terms of an A. P. are given, the series can be completely determined.*

Suppose, x and y are respectively the pth and qth terms of an A. P. Let a be the first term and d the common difference of the A. P. Then,

$$\left. \begin{array}{l} x = a + (p - 1)\ d \\ y = a + (q - 1)\ d \end{array} \right\}$$

These two equations can be solved for a and d and the series is thus completely determined.

Note. *Of the four things,* (i) the first term (a), (ii) the common difference (d), (ii) the number of terms (n) and (iv) the value of the nth term (t_n), *any three being given, we can find the fourth* with the help of the formula (1).

4. Sum of a series in A.P.

Let a be the first term, d the common difference, n the number of terms and l the last term of a series in A.P.

Suppose it is required to find the sum of n terms of this series and let S_n denote this sum.

$$\therefore \quad S_n = a + (a + d) + (a + 2d) + \cdots + (l - d) + l.$$

Writing the series in the reverse order, we get

$$S_n = l + (l - d) + (l - 2d) + \cdots + (a + d) + a.$$

$\therefore$ Adding the corresponding terms, we have

$$2S_n = (a + l) + (a + l) + (a + l) + \cdots\cdots \text{ to } n \text{ terms}$$

$$\therefore \quad \mathbf{S_n = \frac{n}{2}(a+l)} \quad \cdots \quad \cdots \quad (3)$$

Since, l is the nth team. $\qquad \therefore \quad l = a + (n - 1)\,d.$

$$\therefore \quad \mathbf{S_n = \frac{n}{2}\{2a + (n-1)\,d\}.} \quad \cdots \quad (4)$$

5. Properties of A.P.

(i) *If each of the terms of an A.P. be increased or decreased by a constant quantity, the resulting quantities are in A.P. with the same common difference as before.*

(ii) *If each of the terms of an A.P. be multiplied or divided by a constant quantity, the resulting quantities are in A.P. with a common difference equal to that of the given series multiplied or divided by the corresponding constant quantity.*

$$\left.\begin{array}{l} \text{(i) } a + x,\ b + x,\ c + x,\ d + x,\ldots \\[4pt] \text{(ii) } a - x,\ b - x,\ c - x,\ d - x\ldots \\[4pt] \text{(iii) } ax,\ bx,\ cx,\ dx,\ldots \\[4pt] \text{(iv) } \dfrac{a}{x},\ \dfrac{b}{x},\ \dfrac{c}{x},\ \dfrac{d}{x} \end{array}\right\} \quad \cdots \quad (5)$$

Proof : (i) The quantities are in A.P., if $(b + x) - (a + x)$ $= (c + x) - (b + x)$, *i.e.,* if $b - a = c - b$ which is true, since a, b, c are A.P,

(ii) Proof is similar.

(iii) The quantities are in A.P., if $bx - ax = ax - bx$, *i.e.,* if $b - a = c - b$ which is true since a, b, c are in A.P.

(iv) Proof is similar,

6. Arithmetic Mean.

When three quantities are in A.P., the middle term is called the *Arithmetic Mean* (or shortly written as A. M.) of the other two.

If a, m, b are in A.P., we have $m - a = b - m$ *i.e.,* $m = \frac{1}{2}(a + b)$.

Hence, **A. M. of a and b $= \frac{1}{2}(a + b)$.**

Therefore, *the Arithmetic Mean of two given quantities is half their sum.* Thus, the arithmetic mean of two numbers is their *average.*

Of three terms 4, 7, 10 ; 7 is the A. M. between 4 and 10 and obviously, $7 = \frac{1}{2}(4 + 10)$.

The *Arithmetic mean of n quantities* is the sum of all the quantities divided by n. Thus,

$$\left. \begin{aligned} \text{A. M. of } a_1, a_2, \ldots a_n &= \frac{a_1 + a_2 + \cdots + a_n}{n} \\ &= \sum_{r=1}^{n} \frac{a_r}{n}. \end{aligned} \right\} \quad \cdots \quad \cdots \quad (6)$$

Again, when any number of quantities are in A.P., all the terms intermediate between the first and the last are called the *Arithmetic Means* between these two terms.

Thus, 1, $1\frac{1}{2}$, $2\frac{1}{2}$ are 4 A. M.'s between $\frac{1}{2}$ and 3.

To insert a given number of Arithmetic means between two given quantities.

Let a and b be two given quantities and let $m_1, m_2, \ldots m_n$ be n arithmetic means between them.

When n means are inserted between a and b, we shall have an A. P., consisting of $(n+2)$ terms, of which a will be the first term and b the $(n+2)$th term.

Let d denote common difference. Then,

$$b = a + (n+1)d, \text{ whence } d = \frac{b-a}{n+1}.$$

Hence, the reqd. means $m_1, m_2, m_3, \ldots m_n$ are

$$a + \frac{b-a}{n-1}, \quad a + \frac{2(b-a)}{n+1}, \quad a + \frac{3(b-a)}{n+1}, \ldots, \quad a + \frac{n(b-a)}{n+1}.$$

7. Illustrative Examples.

Ex. 1. *Find the A. P. whose 5th and 8th terms are 11 and 17 respectively. Find also its 15th term.*

Let a be the first term d the common difference.

Then, by the question, $\left. \begin{array}{l} a + 4d = 11, \\ a + 7d = 17. \end{array} \right\}$

$$\therefore \quad a = 3, \ d = 2.$$

Hence, the reqd. series is 3, 5, 7, 9, ...
and the 15th term $= a + 14d = 3 + 14 \times 2 = 31$.

Ex. 2. *The sum of n terms of an A. P., whose common difference is 4, is 21. The sum of $2n$ terms is 78. Calculate the first term.*

Let a be the first term. Then, from formula (4),

$$21 = \frac{n}{2}\{2a + (n-1)\,4\} = n\,(a + 2n - 2). \quad \cdots \quad \text{(i)}$$

$$78 = \frac{2n}{2}\{2a + (2n-1)\,4\} = n\,(2a + 8n - 4). \quad \cdots \quad \text{(ii)}$$

$$\therefore \quad \frac{78}{21} = \frac{2a - 8n - 4}{a + 2n - 2}, \text{ which gives } n = 3a - 6. \quad \cdots \quad \text{(iii)}$$

Substituting this value of n in (i), we get

$$21 = (3a - 6)(a + 6a - 12 - 2),$$

i.e., $(a-2)^2 = 1$; *i.e.*, $a - 2 = \pm 1$. $\therefore \quad a = 1$, or, 3.

$\therefore$ From (iii), $n = -3$, or, 3, according as $a = 1$, or, 3.

Since, n is a positive integer, the first value is to be rejected, and the first term is 3.

Ex. 3. *How many terms of the A. P.*

$$2\tfrac{1}{2}, \ 2\tfrac{1}{4}, \ 2\ldots\ldots\ldots$$

must be taken so that their sum may be $13\tfrac{3}{4}$? *Explain the double answer.*

Putting $S = 13\tfrac{3}{4}$, $a = 2\tfrac{1}{2}$, $n = 2\tfrac{1}{4} - 2\tfrac{1}{2} = (-\tfrac{1}{4})$, in formula (4),

we get $\qquad 13\tfrac{3}{4} = \dfrac{n}{2}\{5 + (n-1)(-\tfrac{1}{4})\}$.

$\therefore \quad n^2 - 21n + 110 = 0. \quad \therefore \quad (n-10)(n-11) = 0. \quad \therefore \quad n = 10, \ \text{or}, \ 11.$

$\therefore$ Either 10 or 11 terms may taken and in each case the sum is $13\tfrac{3}{4}$.

The existence of these two answers is explained by the fact that the 11th term is zero ; for $t_{11} = 2\tfrac{1}{2} + (11-1)\cdot(-\tfrac{1}{4}) = 2\tfrac{1}{2} - 2\tfrac{1}{2} = 0$.

Ex. 4. *Sum to n terms :*

$$\frac{n-1}{n} + \frac{n-2}{n} + \frac{n-3}{n} + \ldots\ldots\ldots\ldots$$

Here, $\quad a = \dfrac{n-1}{n}$; $\quad d = \dfrac{n-2}{n} - \dfrac{n-1}{n} = -\dfrac{1}{n}$.

$\therefore$ By formula (4), $S = \dfrac{n}{2}\left\{2\cdot\dfrac{n-1}{n} + (n-1)\left(-\dfrac{1}{n}\right)\right\} = \dfrac{n-1}{2}$.

Ex. 5. *If the nth term of a series in A. P. is always* $3n+1$, *find the sum of n terms of the series.*

Here, $\quad t_n = 3n+1. \quad \therefore \quad t_1$ (*i.e.*, first term) $= 3 + 1 = 4$,

and $\quad l = $ last term (nth term) $= 3n+1$.

$\therefore$ By formula (3), $S = \dfrac{n}{2}\{4 + 3n + 1\} = \dfrac{n}{2}(3n+5)$.

Ex. 6. *Find the series in A. P. whose sum to n terms is* $n(n+2)$.

Let S_n denote the sum to n terms and S_{n-1} the sum to $(n-1)$ terms ; then t_n (nth term) $= S_n - S_{n-1}$

$$= n(n+2) - (n-1)(n+1) = 2n+1.$$

$\therefore$ The series is 3, 5, 7, 9,$\ldots\ldots\ldots\ldots$

Ex. 7. *The sum of n terms of two A.P.'s are in the ratio of* $n-1 : n+1$, *find the ratio of their fifth terms.*

Let $a_1, \ a_2$ be the first terms and $d_1, \ d_2$ the common differences of the two series and $S_1, \ S_2$ denote their sums up to n terms.

$\therefore \qquad S_1 = \tfrac{1}{2}n\{2a_1 + (n-1)\ d_1\}$,

$\qquad\qquad S_2 = \tfrac{1}{2}n\{2a_2 + (n-1)\ d_2\}$.

Since, $\dfrac{S_1}{S_2} = \dfrac{n-1}{n+2}$. $\therefore$ $\dfrac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \dfrac{n-1}{n+1}$. ... (1)

The ratio of the fifth terms of the two series

$$= \frac{a_1 + 4d_1}{a_2 + 4d_2} = \frac{2a_1 + 8d_1}{2a_2 + 8d_2} = \frac{2a_1 + (9-1)d_1}{2a_2 + (9-1)d_2}$$

$$= \frac{9-1}{9+1} \text{ from (1) by putting } n = 9$$

$$= \tfrac{4}{5}, \text{ reqd. ratio.}$$

Ex. 8. *If a, b, c be the pth, qth, rth terms respectively of an A. P.,* then $\qquad a(q-r) + b(r-p) + c(p-q) = 0.$

Let x be the first term and y the common difference.

Then, $\quad a = x + (p-1)y \qquad \cdots \qquad \cdots$ (i)

$\qquad\quad b = x + (q-1)y \qquad \cdots \qquad \cdots$ (ii)

$\qquad\quad c = x + (r-1)y \qquad \cdots \qquad \cdots$ (iii)

$\therefore \quad a - b = (p-q)y \,; \qquad b - c = (q-r)y.$

$\therefore \quad \dfrac{a-b}{b-c} = \dfrac{p-q}{q-r}.$

On cross-multiplication, reqd. result follows.

Ex. 9. *If $\dfrac{1}{a}$, $\dfrac{1}{b}$, $\dfrac{1}{c}$ be in A.P., and $(a+b+c \neq 0)$, prove that*

$$\frac{b+c}{a}, \ \frac{c+a}{b}, \ \frac{a+b}{c} \text{ are in A.P.}$$

Multiply each term of the given A. P. by $a+b+c$.

$\therefore \quad \dfrac{a+b+c}{a}, \ \dfrac{a+b+c}{b}, \ \dfrac{a+b+c}{c}$ are in A.P. by Art. 5.

Subtract 1 from each term of the A. P., then

$$\frac{b+c}{a}, \ \frac{c+a}{b}, \ \frac{a+b}{c} \text{ are in A. P. by Art. 5.}$$

Ex. 10. *In an A.P., prove that the sum of any two terms equidistant from the beginning and the end is equal to the sum of the first and last term.*

Let a be the first term, l the last term, d the common difference, n the number of terms.

The rth term from the beginning $= a + (r-1)d$.

The rth ,, ,, ,, end $= (n-r+1)$th term from the beginning

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad = a + (n-r)d.$

$\therefore$ Reqd. sum $= a + a + (n-1)d = a + l.$

Ex. 11. *Divide 72 into three parts which are in A.P. and such that the product of the first two is 480.*

When in any problem 3 terms are given to be in A.P., it is convenient to take them in the form $x-y$, x, $x+y$.

Then, $(x-y)+x+(x+y)=72$,   *i.e.,* $3x=72$,   *i.e.,* $x=24$
   and $x(x-y)=480$, *i.e.,* $24(24-y)=480$, *i.e.,* $24-y=20$. $\therefore y=4$.

Hence, the reqd. parts are 20, 24, 28.

Ex. 12. *Insert 12 arithmetic means between 4 and 43.*
Let d be the common difference.

Since, the complete series contains 14 terms in all, hence, 43 is the 14th term of the series of which 4 is the 1st term.

$\therefore$ $43=4+13.d.$ $\therefore$ $d=3$.

$\therefore$ The reqd. means are 7, 10, 13,.........40.

Examples (AI)

[*The answer of each example is given at its end within* [ ]]

1. What is the 14th term of an A.P. whose 5th term is 11 and 9th term is 7 ?   [2]

2. Find the common difference of an A.P. whose first term is 5 and the 11th term 125.   [2]

3. The 10th term of a series in A.P. is 23 and 32nd term is 67 ; find the first and common difference.
   [5 ; 2]

4. Which term of the series 9, $10\frac{1}{3}$, $11\frac{2}{3}$, 13,...... is 133 ?   [94th term]

5. Is -447 a term of the series 8, 5, 2,...... ?   [No]

6. If the pth term of an A.P. is q and the qth term is p, show that the mth term is $p+q-m$.

7. Find the sum of each of the following series
 (i) $5+17+29+41+\cdots\cdots$ to 20 terms.   [2380]
 (ii) $20+18+16+\cdots\cdots$ to 12 terms.   [108]

(iii) $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \cdots\cdots$ to 9 terms. $[-1\frac{1}{3}]$

(iv) $1 - 3 - 7 - \cdots\cdots$ to n terms. $[n (3 - 2n)]$

(v) $\sqrt{2} + \sqrt{2}(1 - \sqrt{2}) + \sqrt{2}(1 - 2\sqrt{2}) + \cdots$ to 21 terms.

$[21 (\sqrt{2} - 20)]$

(vi) $5\dot{}3 + 3\dot{}9 + 2\dot{}5 + \cdots$ to 15 terms. $[-67\dot{}5]$

(vii) $n + (n - 1) + (n - 2) + \cdots\cdots$ to n terms. $[\frac{1}{2}n (n + 1)]$

(viii) $(x - y) + (2x - 3y) + (3x - 5y) + \cdots\cdots$ to n terms.

$[\frac{1}{2}n \{(x - 2y) n + x\}]$

(ix) $\dfrac{x^2 - 1}{x} + x + \dfrac{x^2 + 1}{x} + \cdots\cdots$ to n terms.

$$\left[nx + \frac{1}{2}n (n - 3)\frac{1}{x} \right]$$

(x) $(x + y)^2 + (x^2 + y^2) + (x - y)^2 + \cdots\cdots$ to n terms.

$[n (x^2 + y^2) - n (n - 3) xy]$

8. Find *without assuming the summation formula*, the sum of the series $1 + 4 + 7 + \cdots\cdots$ to 21 terms. $[651]$

9. How many terms of the series 2, 4, 6,...... must be taken in order that the sum may be 420 ? $[20]$

10. How many terms of the series $27 + 24 + 21 + \cdots\cdots$ must be taken in order that the sum may be 132 ? Explain the double answer. $[8 \text{ or } 11]$

11. The 12th term of an A.P. is 23 and the sum of the first 22 terms is 484 ; find the sum of the first 30 terms.

$[900]$

12. The sum of n terms of two A.P.'s are in the ratio of $2n + 1 : 2n - 1$; find the ratio of their tenth terms.

$[39 : 37]$

13. The sum of n terms of two A.P.'s are in the ratio of $(13 - 7n) : (3n + 1)$; find the ratio of their first terms and the second terms. $[3 : 2 ; -4 : 5]$

14. If nth terms of a series is always $2n + 3$, find the sum up to n terms. $[n^2 + 4n]$

15. Find the 10th term of a series in A.P., whose sum to n terms is $3n + 4n^2$. $[79]$

16. If the sum of m terms of an A.P., be always to the sum of n terms in the ratio of $m^2 : n^2$ and the first term be unity, show that

$$\frac{m\text{th term}}{n\text{th term}} = \frac{2m-1}{2n-1}.$$

17. If a, b, c are in A.P., show that

(i) $b+c, c+a, a+b$ are in A.P.

(ii) $b+c-a, c+a-b, a+b-c$ are in A.P.

(iii) $1/bc, 1/ca, 1/ab$ are in A.P.

(iv) $(b+c)^2 - a^2, (c+a)^2 - b^2, (a+b)^2 - c^2$ are in A.P.

(v) $a^2(b+c), b^2(c+a), c^2(a+b)$ are in A.P. $(\Sigma bc \neq 0)$.

(vi) $a\left(\dfrac{1}{b} + \dfrac{1}{c}\right), b\left(\dfrac{1}{c} + \dfrac{1}{a}\right), c\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ are in A.P.

(vii) $\dfrac{1}{\sqrt{b} + \sqrt{c}}, \dfrac{1}{\sqrt{c} + \sqrt{a}}, \dfrac{1}{\sqrt{a} + \sqrt{b}}$ are in A.P.

18. If a^2, b^2, c^2 are in A.P., then

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A.P.}$$

19. If $\dfrac{b+c}{a}, \dfrac{c+a}{b}, \dfrac{a+b}{c}$ are in A.P., and $(a+b+c) \neq 0$,

then $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}$ are in A.P.

20. If $\dfrac{a}{b+c}, \dfrac{b}{c+a}, \dfrac{c}{a+b}$ are in A.P., then

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A.P., } (a+b+c \neq 0).$$

21. If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A.P., show that

$$\frac{1}{b-c}, \frac{1}{c-a}, \frac{1}{a-b} \text{ are in A.P.}$$

22. If a, b, c be the sums of p, q, r terms respectively of an A.P., prove that

$$\frac{a}{p}(q-r) + \frac{b}{q}(r-p) + \frac{c}{r}(p-q) = 0.$$

23. Prove that the middle term of an A.P., consisting of an *odd* number of terms is equal to half the sum of the first and last terms.

24. Prove that the sum of the *two* middle terms of an A.P., consisting of an *even* number of terms is equal to the sum of the first and last terms.

25. If t_n denote the nth term of an A.P., and if $t_2 : t_4 = 3 : 7$; find the value of $t_5 : t_9$. ⸱ [$9 : 17$]

26. The sum of three numbers in A.P. is 21 and sum of their squares is 155 ; find the numbers. [5, 7, 9]

27. The base of a right-angled triangle is 12 feet and the three sides are in A.P. Find the length of the hypotenuse.
[15 ft. or 20 ft.]

28. Three numbers are in A.P. ; the difference between the first and the third is 14 and the product of these two is 312 ; find the numbers. [26, 19, 12 *or*, $-12, -19, -26$]

29. The sides of a right-angled triangle are in A.P. ; show that they are proportional to the numbers 3, 4, 5.

30. The sum of five numbers in A.P. is 15 and the sum of their squares is 55 ; find the numbers. [1, 2, 3, 4, 5]

[*Take the numbers as* $a - 2x, a - x, a, a + x, a + 2x$]

31. Insert 22 arithmetic means between 1 and 70.
[4, 7, 10,...67]

32. Insert 36 arithmetic means between $8\frac{1}{2}$ and $2\frac{1}{3}$.
[$8\frac{1}{3}, 8\frac{1}{6},...2\frac{1}{2}$]

33. There are n arithmetic means between 5 and 35, such that the second mean : last mean = 1 : 4 ; find n. [17]

34. Prove that the ratio of the sum of x arithmetic means to the sum of y arithmetic means inserted between any two numbers is $x : y$.

35. If pth term of an A.P. is $1/q$, and qth term is $1/p$, show that the sum of pq term is $\frac{1}{2}(pq + 1)$.

36. If the sum of m terms of a series is to the sum of n terms as $m^2 : n^2$, show that the mth term is to the nth terms as $2m - 1 : 2n - 1$.

37. Find the least value of n for which

$$3 + 6 + 9 + \cdots \text{ to } n \text{ terms exceeds } 1000. \qquad [\ 26\ ]$$

38. A man 50 years old has 8 sons born at equal intervals. The sum of the ages of the father and the sons is 186 years. What is the age of the eldest son, the youngest son being 3 years old ?   [31 years]

39. If 100 stons be placed in a straight line, exactly a yard apart, the first being one yard from the basket. What distance will a person go, who gathers them singly, returning with each to the basket ?   [5 miles 1300 yds.]

40. A man undertakes to pay off a debt of Rs. 65 by monthly instalments ; he pays Rs. 2 in the first month and continually increases the instalment in every subsequent month by Re. 1. In what time will the debt be cleared up ?
                                            [10 months]

41. A class consists of a number of boys whose ages are in A.P., the common difference being 4 months. If the youngest boy is just eight years old and the sum of the ages is 168 years, find the number of boys in the class.   [16]

42. (i) If the nth term of a series $= b + nc$, show that the series is in A.P.

    (ii) If the sum to n terms of a series $= bn^2 + cn$, show that the series is in A.P.

43. A series of n terms in A.P. ; show that

    (i) if n be *odd*, the sum of the series is n times the middle term.

    (ii) If n be *even*, the sum is equal to n times half the sum of two middle terms.

44. If s_1, s_2, s_3 be the sums of n terms of three series in A.P., whose first terms are the same and whose common differences are in A.P., show that s_1, s_2, s_3 are in A.P.

45. If s_n denote the sum of n terms of an A.P., prove that

$$qr\,(q-r)\,s_{pm} + rp\,(r-p)\,s_{qm} + pq\,(p-q)\,s_{rm} = 0.$$

46. The sums of the first n_1, n_2, n_3 terms of the same arithmetic series are S_1, S_2, S_3 respectively ; show that

$$\frac{S_1}{n_1}\left(n_2 - n_3\right) + \frac{S_2}{n_2}\left(n_3 - n_1\right) + \frac{S_3}{n_3}\left(n_1 - n_2\right) = 0.$$

8. Sum of the natural numbers.

The numbers 1, 2, 3,... are called the *natural numbers*.

(*a*) *Sum of the first n natural numbers.*

Let $s_n = 1 + 2 + 3 + \cdots + n$

$$= \frac{n}{2}\{2 + (n-1).1\}, \text{ since here c.d.} = 1,$$

[*by formula (4)*]

$$\therefore \quad S_n = \frac{n\,(n+1)}{2}. \qquad \qquad \cdots \qquad \cdots \quad (8)$$

(*b*) *Sum of the first n odd natural numbers.*

Let $s_n = 1 + 3 + 5 + \cdots$ to n terms

$$= \frac{n}{2}\{2 + (n-1)\,2\} \qquad \qquad [\ by\ formula\ (4)\ ]$$

$$= \frac{n}{2} \cdot 2n = n^2.$$

$$\therefore \quad S_n = n^2. \qquad \qquad \cdots \qquad \cdots \quad (9)$$

(*c*) *Sum of the first n even natural numbers.*

Let $s_n = 2 + 4 + 6 + \cdots$ to n terms

$$= 2(1 + 2 + 3 + \cdots \text{ to } n \text{ terms})$$

$$= 2 \cdot \frac{n\,(n+1)}{2}.$$

$$\therefore \quad S_n = n\,(n+1). \qquad \qquad \cdots \qquad \cdots \quad (10)$$

(*d*) *Sum of the square of the first n natural numbers.*

Let $s_n = 1^2 + 2^2 + 3^2 + \cdots + n^2$.

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We have $n^3 - (n-1)^3 = 3n^2 - 3n + 1$.

Hence, putting $n = 1, 2. 3, \ldots\ldots$ in succession, we have

$$1^3 - 0^3 = 3.1^2 - 3.1 + 1$$
$$2^3 - 1^3 = 3.2^2 - 3.2 + 1$$
$$\cdots \qquad \cdots \qquad \cdots$$
$$n^3 - (n-1)^3 = 3n^2 - 3n + 1.$$

$\therefore$ Adding $n^3 = 3(1^2 + 2^2 + \cdots + n^2) - 3(1 + 2 + \cdots + n) + n$
$$= 3S_n - 3.\tfrac{1}{2}n(n+1) + n,$$

$$\therefore \quad 3S_n = n^3 - n + \tfrac{3}{2}n(n+1)$$
$$= n(n^2 - 1) + \tfrac{3}{2}n(n+1)$$
$$= n(n+1)(n-1+\tfrac{3}{2}) = \tfrac{1}{2}n(n+1)(2n+1).$$

$$\therefore \quad \mathbf{S_n = \frac{n(n+1)(2n+1)}{6}}. \qquad \cdots \quad (11)$$

(e) *Sum of the cubes of the first n natural numbers.*

Let $\qquad S_n = 1^3 + 2^3 + \cdots + n^3$.

We have $n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1$.

Hence, putting $n = 1, 2, 3, \ldots$ in succession, we have

$$1^4 - 0^4 = 4.1^3 - 6.1^2 + 4.1 - 1$$
$$2^4 - 1^4 = 4.2^3 - 6.2^3 + 4.2 - 1$$
$$\cdots \qquad \cdots \qquad \cdots \qquad \cdots$$
$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1$$
$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4.n - 1.$$

$\therefore$ Adding $n^4 = 4(1^3 + 2^3 + \cdots + n^3) - 6(1^2 + 2^2 + \cdots + n^2)$
$$+ 4(1 + 2 + \cdots + n) - n$$
$$= 4S_n - 6.\tfrac{1}{6}n(n+1)(2n+1) + 4.\tfrac{1}{2}n(n+1) - n.$$

$$\therefore \quad 4S_n = (n^4 + n) + n(n+1)(2n+1) - 2n(n+1)$$
$$= n^2(n+1)^2, \text{ on simplification.}$$

$$\therefore \quad \mathbf{S_n = \left\{ \frac{n(n+1)}{2} \right\}^2}. \qquad \cdots \quad (12)$$

Cor. Thus, $1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$.

Note. For sake of brevity the Greek letter Σ (*sigma*) is used before a letter to denote the sum of a number of terms of that type. Thus,

$$1 + 2 + 3 + \cdots + n \text{ is denoted by } \Sigma n$$

$$1.2 + 2.3 + \cdots \cdots n\,(n+1) \text{ is denoted by } \Sigma n\,(n+1).$$

9. Illustrative Examples.

Ex. 1. *Sum to n terms the series :*

$$1.2 + 2.3 + \cdots + n\,(n+1).$$

Here, $\qquad t_n = n\,(n+1) = n^2 + n.$

$\therefore \quad \Sigma n\,(n+1) = \Sigma n^2 + \Sigma n$

$$= \frac{n\,(n+1)(2n+1)}{6} + \frac{n\,(n+1)}{2} \text{ by formula (11) and (8)}$$

$$= \frac{n\,(n+1)}{2}\left(\frac{2n+1}{3} + 1\right)$$

$$= \tfrac{1}{3}n\,(n+1)(n+2).$$

Ex. 2. *Sum the series :*

$$1 + (1+2) + (1+2+3) + (1+2+3+4) + \cdots \text{ to n terms.}$$

Here t_n is evidently $(1+2+3+\cdots+n) = \tfrac{1}{2}\,(n+1) = \tfrac{1}{2}n^2 + \tfrac{1}{2}n.$

$\therefore \quad$ Sum of the given series $= \tfrac{1}{2}\Sigma n^2 + \tfrac{1}{2}\Sigma n$

$$= \frac{1}{2} \cdot \frac{n\,(n+1)(2n+1)}{6} + \frac{1}{2} \cdot \frac{n\,(n+1)}{2}, \text{ by formula (11) and (8)}$$

$$= \tfrac{1}{6}n\,(n+1)(n+2), \text{ on simplification.}$$

Ex. 3. *Sum the series :*

$$1 + 3 + 6 + 10 + 15 + \cdots \text{ to n terms.}$$

Here the series though not in A. P. the successive difference of the terms (namely 2, 3, 4, 5, etc.) *are in A. P.* To sum the series we should proceed in the following way.

Let $\qquad S_n = 1 + 3 + 6 + 10 + 15 + \cdots + t_{n-1} + t_n.$

Again $\quad S_n = \qquad 1 + 3 + 6 + 10 + \cdots + t_{n-2} + t_{n-1} + t_n.$

Now, subtracting, $0 = 1 + (3-1) + (6-3) + (10-6) + \cdots + (t_n - t_{n-1}) - t_n$

$$= 1 + 2 + 3 + 4 + \cdots \text{ to n terms} - t_n.$$

On transposition,

$\therefore$ t_n (nth term of the given series) $= 1 + 2 + 3 + \cdots$ to n terms
$$= \tfrac{1}{2}n\,(n+1) = \tfrac{1}{2}n^2 + \tfrac{1}{2}n.$$

$\therefore$ $S_n = \tfrac{1}{2}\Sigma n^2 + \tfrac{1}{2}\Sigma n$
$$= \tfrac{1}{6}n\,(n+1)(n+2)\ \text{as in Ex. 2.}$$

Ex. 4. *Sum the series :*

$$1 + 2 - 3 + 4 + 5 - 6 + 7 + 8 - 9 + \cdots \text{ to } (3n+1)\ terms.$$

nth term of the series $1, 4, 7,\ldots = 1 + (n-1)\,3 = 3n - 2$

     ,,   ,, $2, 5, 8,\ldots = 2 + (n-1)\,3 = 3n - 1$

     ,,   ,, $3, 6, 9,\ldots = 3 + (n-1)\,3 = 3n.$

$\therefore$ Sum of 3 terms ending at $3n$th terms of the given series
$$= 3n - 2 + 3n - 1 - 3n = 3n - 3.$$

$(3n+1)$th term of the given series $= 3n + 1.$

$\therefore$ Sum to $3n$ terms $= 3\Sigma n - \Sigma 3$
$$= 3 \cdot \frac{n\,(n+1)}{2} - 3n = \frac{3}{2}\,n\,(n-1).$$

$\therefore$ Sum to $(3n+1)$ terms $= \tfrac{3}{2}n\,(n-1) + 3n + 1 = \tfrac{1}{2}\,(3n^2 + 3n + 2).$

Examples (AII)

Sum to n terms the following series :

1. $1.1 + 2.3 + 3.5 + 4.7 + \cdots$ $[\ \tfrac{1}{6}n\,(n+1)(4n-1)\ ]$
2. $2.1 + 3.3 + 4.5 + \cdots$ $[\ \tfrac{1}{6}n\,(4n^2 + 9n - 1)\ ]$
3. $1.3 + 3.5 + 5.7 + 7.9 + \cdots$ $[\ \tfrac{1}{3}n\,(4n^2 + 6n - 1)\ ]$
4. $3.8 + 6.11 + 9.14 + \cdots$ $[\ 3n\,(n+1)(n+3)\ ]$
5. $1.2.3 + 2.3.4 + 3.4.5 + 4.5.6 + \cdots$

                                  $[\ \tfrac{1}{4}n\,(n+1)(n+2)(n+3)\ ]$

6. $1.3.5 + 3.5.7 + 5.7.9 + \cdots$ $[\ n\,(n+2)(2n^2 + 4n - 1)\ ]$
7. $1.3.5 + 2.4.6 + 3.5.7 + \cdots$ $[\ \tfrac{1}{4}n\,(n+1)(n+4)(n+5)\ ]$
8. $1^2 + 3^2 + 5^2 + 7^2 + \cdots$ $[\ \tfrac{1}{3}n\,(4n^2 - 1)\ ]$
9. $2^2 + 5^2 + 8^2 + \cdots$ $[\ \tfrac{1}{2}n\,(6n^2 + 3n - 1)\ ]$
10. $1.2^2 + 2.3^2 + 3.4^2 + \cdots$ $[\ \tfrac{1}{12}n\,(n+1)(n+2)(3n+5)\ ]$
11. $2.1^2 + 3.2^2 + 4.3^2 + \cdots$ $[\ \tfrac{1}{12}n\,(n+1)(n+2)(3n+1)\ ]$

12. $1^3 + 3^3 + 5^3 + 7^3 + \cdots$ $\qquad$ $[\ n^2(2n^2 - 1)\ ]$

13. $1 + (1 + 3) + (1 + 3 + 5) + \cdots$ $\qquad$ $[\ \frac{1}{6}n(n+1)(2n+1)\ ]$

14. $1^2 + (1^2 + 2^2) + (1^2 + 2^2 + 3^2) + (1^2 + 2^2 + 3^2 + 4^2) + \cdots$

$$\left[\frac{n(n+1)^2(n+2)}{12}\right]$$

15. $n \cdot 1 + (n-1) \cdot 2 + (n-2) \cdot 3 + \cdots + 1 \cdot n.$

$$[\ \tfrac{1}{6}n(n+1)(n+2)\ ]$$

16. $3 + 7 + 13 + 21 + \cdots$ $\qquad$ $\cdot \left[\dfrac{n(n^2 + 3n + 5)}{3}\right]$

17. $5 + 6 + 9 + 14 + 21 + \cdots$ $\qquad$ $[\ \frac{1}{6}n(2n^2 - 3n + 31)\ ]$

18. $2 + 11 + 28 + 53 + 86 + \cdots$ $\qquad$ $[\ \frac{1}{6}n(8n^2 + 3n + 1)\ ]$

19. Sum to $(2n + 1)$ terms the series

$1 - 2 + 3 - 4 + \cdots$ $\qquad$ $[\ n + 1\ ]$

20. Find the sum of n terms of the series whose nth term is $6n^2 - 2n + 1$. $\qquad$ $[\ n(2n^2 + 2n + 1)\ ]$

21. Show that $1 - 3 + 5 - 7 + \cdots$ to n terms $= (-1)^{n+1}\, n.$

22. Find the sum to n terms of $8 - 11 + 14 - 17 + \cdots$

$[\ -1\frac{1}{2}n$ if n even ; $1\frac{1}{2}n + 6\frac{1}{2}$ if n odd $]$

23. Find the sum of $1^2 - 2^2 + 3^2 - 4^2 + \cdots$ to $2n$ terms.

$[\ -n(2n + 1)\ ]$

24. If S_n denote the sum of n terms of A.P., show that

$$S_{n+3} - 3S_{n+2} + 3S_{n+1} - S_n = 0.$$

SEC. B.—GEOMETRICAL PROGRESSION

10. Definition.

A series of quantities are said to be in Geometrical Progression when the ratio of any term to the preceding one is the same throughout the series.

Thus, $a, b, c, d,\ldots$ are in Geometrical Progression if $\dfrac{b}{a} = \dfrac{c}{b} = \dfrac{d}{c} =$ etc.

The ratio of each term to the preceding is called the *common ratio*.

Thus, the series of the quantities

(i) $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\ldots$ (ii) $3, -1, \frac{1}{3}, -\frac{1}{9}, \cdots$ are in Geometrical Progression, the common ratio of (i) being $\frac{1}{2}$ and that of (ii) being $-\frac{1}{3}$.

Note 1. For the sake of convenience the words 'Geometrical Progression' and 'common ratio' are abbreviated as *G.P.* and *C.R.*

Note 2. From the definition of a G.P. it is clear that each term is formed from the preceding by multiplying it by a constant factor, the common ratio.

Note 3. It should be noted that the *terms of a G.P. are in continued proportion*.

11. The nth term.

Let a be the first term and r the common ratio. Then by definition,

$$2\text{nd term} = ar\ = ar^{2-1}$$
$$3\text{rd term} = ar^2 = ar^{3-1}$$
$$4\text{th term} = ar^3 = ar^{4-1}$$
$$\therefore \quad \textbf{nth term} \qquad = \textbf{ar}^{\textbf{n}-\textbf{1}} \qquad \cdots \quad (1)$$

Thus, *we can write down any term of a G.P. whose first term and common ratio are given.*

Thus, if the first term *i.e.*, $a = 5$ and the common ratio $r = 2$, then the 9th term of the series *i.e.*,

$$t_9 = 5 \cdot 2^{9-1} = 5 \cdot 2^8 = 5 \times 256 = 1280.$$

If a be the first term, r the common ratio, n the number of terms and l the last term (*i.e.*, the *n*th term), then

$$l = \textbf{ar}^{\textbf{n}-\textbf{1}} \qquad \cdots \quad (2)$$

Cor. *When any two terms of a G.P. are given, the series can be completely determind.*

Suppose, x and y are respectively the pth and qth terms of a G.P.

Let a be the first term and r the common ratio of the G.P. Then $x = ar^{p-1}$, $x = ar^{q-1}$. From these two equations we can easily determine a and r and hence the whole geometrical series.

12. Sum of a series in G.P.

Let a be the first term and r the common ratio of a series in G.P. Suppose it is required to find the sum of n terms of this series and let S_n denote the sum.

$$\therefore \quad S_n = a + ar + ar^2 + \cdots + ar^{n-1}.$$

Multiplying by r, we get

$$(S_n) \cdot r = ar + ar^2 + \cdots + ar^{n-1} + ar^n.$$

Hence, by subtraction,

$$S_n - S_n \cdot r = a - ar^n.$$

$$\therefore \quad \mathbf{S_n = a \cdot \frac{1 - r^n}{1 - r} = a \cdot \frac{r^n - 1}{r - 1}.} \qquad \cdots \quad (3)$$

Cor. If l be the last term of the series, then $l = ar^{n-1}$.

$\therefore$ Formula (3) can be written as

$$\mathbf{S_n = \frac{a - rl}{1 - r} = \frac{rl - a}{r - 1}.} \qquad \cdots \quad (4)$$

Note. The formula (3) fails if $r = 1$; for in this case the formula assumes the form $\frac{0}{0}$ which is meaningless.

However, $S_n = a + a + a + \cdots$ to n terms $= na$.

13. Geometric Mean.

When three quantities are in G.P., the Middle one is called the geometric mean (G.M.) of the other two.

If $a, m. b$ are in G.P., we have by definition,

$$\frac{m}{a} = \frac{b}{m}. \qquad \therefore \quad m^2 = ab, \; i.e., \; m = \sqrt{ab}.$$

Therefore, *the geometric mean between two given quantities is the square root of their product*. Thus, the geometric mean of two quantities is the same as the *mean proportional*.

Of the three terms 2, 4, 8; 4 is the geometric mean of 2 and 8 and 4 is obviously equal to $\sqrt{2 \times 8} = 4$.

The *geometrical mean of n quantities* is the nth root of their product. Thus

G.M. of $a_1, a_2, \ldots a_n = \sqrt[n]{a_1 a_2 \ldots \ldots a_n}.$

Again when any number of quantities are in G.P., all the terms intermediate between the first and last are called *Geometric Means* between these two terms.

Thus, $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}$ are the 4 geometric means between 1 and $\frac{1}{32}$.

To insert a given number of geometric means between two given quantities.

Let a and b be two given quantities and let $m_1, m_2, \ldots m_n$ be n geometric means between them.

When n means are inserted between a and b, we shall have a G.P. consisting of $(n + 2)$ terms of which a will be the first term and b the $(n + 2)$th term.

Let r be the common ratio ; then

$$b = ar^{n+1}, \text{ whence } r = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}.$$

Hence the reqd. means $m_1, m_2, \ldots m_n$ are $ar, ar^2, ar^3, \ldots ar^n$,

i.e., $a.\left(\frac{b}{a}\right)^{\frac{1}{n+1}}, \quad a.\left(\frac{b}{a}\right)^{\frac{2}{n+1}}, \quad a.\left(\frac{b}{a}\right)^{\frac{3}{n+1}}, \ldots a.\left(\frac{b}{a}\right)^{\frac{n}{n+1}}.$

14. Properties of a G.P.

(i) *If all the terms of a G.P. be multiplied (or divided) by the same quantity, the products (or quotients) will be in G.P.*

(ii) *The reciprocals of the terms of a G.P. are also in G.P.*

(iii) *If each term of a G.P. be raised to the same power, the resulting terms form a G.P.*

Thus. *if a, b, c, d,... be in G.P.,*

(i) $ax, bx, cx, dx, \ldots$ are in G.P.

$a/x, b/x, c/x, d/x \ldots$ are in G.P.

(ii) $\dfrac{1}{a}, \dfrac{1}{b}, \dfrac{1}{c}, \dfrac{1}{d}, \ldots$ are in G.P.

(iii) $a^m, b^m, c^m, d^m, \ldots$ are in G.P.

The above results follow at once from the definition of a G.P.

Thus, for (ii), $\dfrac{1}{a}$, $\dfrac{1}{b}$, $\dfrac{1}{c}$, $\cdots$ will be in G.P., if

$$\frac{1}{b} \div \frac{1}{a} = \frac{1}{c} \div \frac{1}{b}, \quad i.e., \text{ if } \frac{a}{b} = \frac{b}{c}, \quad i.e., \text{ if } \frac{b}{a} = \frac{c}{b},$$

which is true since a, b, c are in G.P.

15. Illustrative Examples.

Ex. 1. *In a G. P., the sum of the first two terms is $\frac{4}{3}$ and the sum of the next two is 12. Write down the series.*

Let a be the first term and r the common ratio.

Then, $\qquad a + ar = \frac{4}{3}$, $\qquad i.e., \quad a(1+r) = \frac{4}{3}$ $\qquad \cdots$ (1)

and $\qquad ar^2 + ar^3 = 12$, $\quad i.e., \quad ar^2(1+r) = 12$ $\qquad \cdots$ (2)

$\therefore$ by division, $r^2 = 12 \times \frac{3}{4} = 9$. $\qquad \therefore \quad r = \pm 3$.

From (1), when $r = 3$, $4a = \frac{4}{3}$, $\qquad \therefore \quad a = \frac{1}{3}$,

and when $r = -3$, $-2a = \frac{4}{3}$. $\qquad \therefore \quad a = -\frac{2}{3}$.

Hence, the series is either

$$\tfrac{1}{3}, 1, 3, 9, 27, \cdots\cdots\cdots$$
$$\text{or,} \quad -\tfrac{2}{3}, 2, -6, 18, -54, \cdots\cdots\cdots$$

Ex. 2. *Find the sum to n terms of the following series :*
$$1 - \tfrac{1}{2} + \tfrac{1}{4} - \tfrac{1}{8} + \tfrac{1}{16} - + \cdots\cdots\cdots$$

Here the first term $(i.e., a) = 1$; $\quad c.r.$ $(i.e., r) = -\frac{1}{2}$,

by formula (3), $S_n = \dfrac{1-(-\frac{1}{2})^n}{1-(-\frac{1}{2})} = \dfrac{2}{3}\left\{1 + (-1)^{n+1} \cdot \dfrac{1}{2^n}\right\}$.

Ex. 3. *How many terms of the series, $1 + 2 + 4 + 8 \cdots\cdots$, must be taken to make 8191 ?*

Here, $a = 1$, $r = 2$, $S_n = 8191$; to find n.

$$S_n, \quad i.e., \quad 8191 = \frac{2^n - 1}{2 - 1} = 2^n - 1. \qquad \therefore \quad 2^n = 81 = 8192 = 2^{13}.$$

$\therefore \quad n = 13$.

Ex. 4. *If a, b, c, d are in G.P., then*
$$a^2 + b^2, \ b^2 + c^2, \ c^2 + d^2 \ are \ in \ G.P.$$

Since, a, b, c, d are in G. P.,

$$\therefore \quad \frac{b}{a} = \frac{c}{b} = \frac{d}{c}.$$

$$\therefore \quad \frac{b^2}{a^2} = \frac{c^2}{b^2} = \frac{d^2}{c^2}. \qquad \therefore \quad \text{each} = \frac{b^2 + c^2}{a^2 + b^2} = \frac{c^2 + d^2}{b^2 + c^2},$$

i.e., $a^2 + b^2$, $b^2 + c^2$, $c^2 + d^2$ are in G.P.

Otherwise :

Let r denote the common ratio of the series a, b, c, d,...

then $\quad b = ar$, $c = ar^2$, $d = ar^3$.

$\therefore \quad a^2 + b^2 = a^2 + a^2 r^2 = a^2 \,(1 + r^2)$; $b^2 + c^2 = a^2 r^2 + a^2 r^4 = a^2 \,(1 + r^2).r^2$;

$c^2 + d^2 = a^2 r^4 + a^2 r^6 = a^2 \,(1 + r^2).r^4.$

Hence, $a^2 + b^2$, $b^2 + c^2$, $c^2 + d^2$ are in G.P. of which $a^2 \,(1 + r^2)$ is the first term and r^2 the common ratio.

Ex. 5. *Sum the following series to n terms*

$$1 + 3 + 7 + 15 + \cdots\cdots$$

[*Here the series though not in G.P., the difference of the terms are in G.P.*]

Let $\quad S_n = 1 + 3 + 7 + 15 + \cdots\cdots\cdots + t_{n-1} + t_n.$

Again, $\quad S_n = \qquad 1 + 3 + 7 + \cdots\cdots\cdots + t_{n-2} + t_{n-1} + t_n.$

$\therefore$ By subtraction, $0 = 1 + (3 - 1) + (7 - 3) + (15 - 7) \cdots + (t_n - t_{n-1}) - t_n.$

$$= (1 + 2 + 2^2 + 2^3 + \cdots\cdots \text{ to } n \text{ terms}) - t_n.$$

$\therefore \quad t_n = 1 + 2 + 2^2 + 2^3 + \cdots\cdots$ to n terms

$$= 1. \frac{2^n - 1}{2 - 1} = 2^n - 1 \qquad [\,by\ formula\ (3)\,]$$

$\left. \begin{array}{l} \therefore \quad t_1 = 2 - 1 \\ \quad t_2 = 2^2 - 1 \\ \quad t_3 = 2^3 - 1 \\ \therefore \quad t_n = 2^n - 1 \end{array} \right\}$
$\begin{array}{l} \therefore \quad S_n = (2 + 2^2 + \cdots\cdots + 2^n) - n \\ \quad = 2\,(1 + 2 + \cdots\cdots \text{ to } n \text{ terms}) - n \\ \quad = 2. \dfrac{2^n - 1}{2 - 1} - n = 2\,(2^n - 1) - n \\ \quad = 2^{n+1} - 2 - n. \end{array}$

Ex. 6. *Sum the series to n terms*

$$4 + 44 + 444 + 4444 + \cdots\cdots\cdots$$

$$S_n = 4\,[1 + 11 + 111 + \cdots\cdots \text{ to } n \text{ terms}]$$

$$= \tfrac{4}{9}\,[9 + 99 + 999 + 9999 + \cdots\cdots \text{ to } n \text{ terms}]$$

$$= \tfrac{4}{9}\,[(10 - 1) + (100 - 1) + (1000 - 1) + \cdots\cdots \text{ to } n \text{ terms}]$$

$$= \tfrac{4}{9}\,[(10 + 100 + 1000 + \cdots\cdots \text{ to } n \text{ terms}) - n]$$

$$= \frac{4}{9} \left\{ \frac{10\,(10^n - 1)}{10 - 1} - n \right\} = \frac{40}{81}\,(10^n - 1) - \frac{4n}{9}.$$

Ex. 7. *Sum the series* $\cdot9+\cdot99+\cdot999+\cdots\cdots$ *to n terms.*

$$\text{The series} = \frac{9}{10}+\frac{99}{100}+\frac{999}{1000}+\cdots\cdots \quad \text{to } n \text{ terms}$$

$$= \left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\cdots\cdots \quad \text{to } n \text{ terms}$$

$$= n-\left(\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\cdots\cdots\right)$$

$$= n-\frac{\dfrac{1}{10}\left(1-\dfrac{1}{10^n}\right)}{1-\dfrac{1}{10}} \qquad [\textit{ by formula (3) }]$$

$$= n-\frac{1}{9}\left(1-\frac{1}{10^n}\right)\cdot \qquad [\textit{ On simplification }]$$

Ex. 8. *Sum to n terms*

$$1+3.6+5.6^2+7.6^3+\cdots\cdots$$

[The rth term of the series consists of the product of 2 factors : one the rth term of an A.P, and the other rth term of a G.P. Such a series may be termed *Arithmetico-Geometrical series.*]

$$t_n=\{1+(n-1)\,2\}.6^{n-1}=(2n-1)\,6^{n-1}.$$

$$S_n=1+3.6+5.6^2+\cdots\cdots+(2n-1)\,6^{n-1}$$

$$\underline{6S_n=\qquad 1.6+3.6^2+\cdots\cdots+(2n-3)\,6^{n-1}+(2n-1)\,6^n}$$

by subtraction,

$$-5S_n=1+2.6+2.6^2+\cdots\cdots+2.6^{n-1}-(2n-1)\,6^n$$

$$=1+2.6\,[1+6+\cdots\cdots \text{ to } (n-1) \text{ terms }]-(2n-1)\,6^n$$

$$=1+2.6\cdot\frac{6^{n-1}-1}{6-1}-(2n-1)\,6^n.$$

$$\therefore\quad S_n=\tfrac{1}{5}\,(2n-1)\,6^n-\tfrac{12}{25}6^{n-1}+\tfrac{7}{25}=\frac{(10n-7)\,6^n+7}{25}.$$

Ex. 9. *Find the sum of the series in the 10th group of*

$$1+(3+3^2)+(3^3+3^4+3^5)+\cdots\cdots$$

The number of terms of the series $1, 3, 3^2,\ldots\ldots$ in the 10 groups is the sum of the series $1+2+3+\cdots\cdots$ to 10 terms $=\tfrac{10}{2}\,(1+10)=55$. Similarly, the number of terms in the 9 groups $=\tfrac{9}{2}\,(1+9)=45$.

If S_n denote the sum of the series $1+3+3^2+3^3+\cdots$ to n terms, the sum of the series in the 10th group

$$=S_{55}-S_{45}=\frac{3^{55}-1}{3-1}-\frac{3^{45}-1}{3-1}=\frac{3^{45}}{2}\,(3^{10}-1).$$

Ex. 10. *The sum of three terms in G.P. is 73 and their product is 512 ; find the terms.*

When in any problem 3 terms are given to be in G. P., it is convenient to take them as $\frac{x}{y}$, x, xy.

$$\therefore \quad \frac{x}{y} + x + xy = 70, \quad i.e., \; x\left(\frac{1}{y} + 1 + y\right) = 73 \quad \cdots \quad \cdots \quad (1)$$

Also, $\frac{x}{y} \cdot x . xy = 512$, $\quad i.e., \; x^3 = 512 = 8^3$. $\therefore x = 8$. $\cdots$ (2)

$\therefore$ From (1) and (2), $\frac{1}{y} + 1 + y = \frac{73}{8}$, $\quad i.e., \quad 8y^2 - 65y + 8 = 0$.

$\therefore (y - 8)(8y - 1) = 0$. $\quad \therefore \; y = 8 \; \text{or} \; \frac{1}{8}$,

Thus, the terms are $\frac{8}{8}$, 8, 8×8, $\quad i.e., \quad 1, \; 8, \; 64$ $\Big\}$
　　　　or, 8.8, 8, $8.\frac{1}{8}$. $\quad i.e., \quad 64, \; 8, \; 1$

Ex. 11. *Show that the Arithmetic mean of two positive unequal quantities a and b is greater than their geometric mean.*

The A.M. of a, b is, by Art. 6, $= \frac{1}{2}(a + b)$

　　　　and the G.M. of a, b is, by Art. 13, $= \sqrt{(ab)}$.

Now, $\qquad \frac{1}{2}(a + b) \; > \; \sqrt{(ab)}$

　　　　if $(a + b) \; > \; 2\sqrt{(ab)}$

i.e., if $(a + b)^2 \; > \; 4ab$

i.e., if $(a + b)^2 \; - \; 4ab > 0$

i.e., if $(a - b)^2 \; > \; 0$, which is true.

Hence the result.

Ex. 12. *If A.M. of two numbers be twice their G.M., show that the numbers are as $2 + \sqrt{3} : 2 - \sqrt{3}$.*

Let x and y be two numbers.

$\therefore \quad \frac{1}{2}(x + y) = 2\sqrt{(xy)}$, $\quad i.e., \quad \frac{1}{4}(x + y)^2 = 4\,(xy)$.

$\therefore \quad \frac{(x + y)^2}{4xy} = 4$.

$\therefore$ By Dividendo, $\frac{(x + y)^2}{(x - y)^2} = \frac{4}{3}$, $\quad i.e., \quad \frac{x + y}{x - y} = \frac{2}{\sqrt{3}}$.

$\therefore$ By Componendo and Dividendo, $\frac{x}{y} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}}$.

Examples (BI)

1. The 3rd and 6th terms of a series in G.P. are 3 and 81 respectively ; find the first term and the common ratio. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad [\frac{1}{3}, \, 3\,]$

2. The 2nd and 3rd terms of a G.P. are together $= 24$ and the next two $= 216$; determine the first term.

$$[\text{ 2 or 4 }]$$

3. Sum the following series :—

(i) $1 + 2 + 4 + 8 + \cdots$ to 15 terms. $[32767]$

(ii) $128 + 64 + 32 + \cdots$ to 10 terms. $[255\frac{3}{4}]$

(iii) $1 - 3 + 9 - 32 + \cdots$ to 9 terms. $[4921]$

(iv) $\dfrac{1}{\sqrt{2}} - 1 + \sqrt{2} - \cdots$ to 10 terms.

$$\left[-\tfrac{31}{2} (2 - \sqrt{2}) \right]$$

(v) $\dfrac{x+y}{x+y}, 1, \dfrac{x-y}{x+y}, \cdots$ to n terms.

$$\left[-\frac{(x+y)^2}{2y(x-y)} \left\{ \left(\frac{x-y}{x+y} \right)^n - 1 \right\} \right]$$

(vi) $a - ar + ar^2 - ar^3 + \cdots$ to n terms.

$$\left[a \cdot \frac{1 + (-1)^{n+1} \cdot r^n}{1+r} \right]$$

(vii) $6 + 12 + 24 + \cdots + 768$. $[1530]$

(viii) $1 - 2 + 4 - 8 + \cdots$ to $2n$ terms. $[\tfrac{1}{3} \{1 - 2^{2n}\}]$

4. Find without assuming the summation formula the sum of $1 + 3 + 9 + \cdots$ to n terms. $[\tfrac{1}{2} (3^n - 1)]$

5. How many terms of the series $64, 32, 16, 8, \ldots$ must be taken so that the sum may be $127\frac{1}{2}$? $[8]$

6. Find the least value of n for which

$$1 + 3 + 3^2 + \cdots + 3^{n-1} > 1000.$$

$$[7]$$

7. If a, b, c, d are in G.P., show that

(i) $a + b, b + c, c + d$ are in G.P.

(ii) $a^2 - b^2, b^2 - c^2, c^2 - d^2$ are in G.P.

(iii) $a^2 + b^2 + c^2, \quad ab + bc + cd, \quad b^2 + c^2 + d^2$ are also in G.P.

(iv) $\dfrac{1}{a^2 + b^2}, \dfrac{1}{b^2 + c^2}, \dfrac{1}{c^2 + d^2}$ are in G.P.

(v) $(a+b)^2$, $(b+c)^2$, $(c+d)^2$ are in G.P.

(vi) $(a-b)^2$, $(b-c)^2$, $(c-d)^2$ are in G.P.

(vii) $(a^3+b^3)^{-1}$, $(b^3+c^3)^{-1}$, $(c^3+d^3)^{-1}$ are in G.P.

8. (i) If a, b, c be in G.P., show that
$$a^2 b^2 c^2 \left(\frac{1}{a^3}+\frac{1}{b^3}+\frac{1}{c^3}\right)=a^3+b^3+c^3.$$

(ii) If a, b, c, d be in G.P., show that
$$(b-c)^2+(c-a)^2+(d-b)^2=(a-d)^2.$$

9. If x, y, z be the pth, qth and rth terms respectively of a series in G.P., show that
$$x^{q-r}\, y^{r-p}\, z^{p-q}=1.$$

10. If x, y, z be in G.P. and p, q, r be in A.P., then
$$x^{q-r}\, y^{r-p}\, z^{p-q}=1.$$

11. If S_n denote the sum of n terms of a G.P., whose first term is a and common ratio is r, prove that
$$S_n (S_{3n}-S_{2n})=(S_{2n}-S_n)^2.$$

12. If S is the sum of n terms of a G.P., P is the product of the terms, R is the sum of the reciprocals of the terms, prove that
$$P^2=\left(\frac{S}{R}\right)^n.$$

13. If P be the continued product of a series of terms in G.P., show that
$$P^2=(ab)^n,$$
where a and b are first and last terms.

14. In a G.P., prove that the product of any two terms equidistant from the beginning and end is constant and is equal to the product of the first and last terms.

15. Prove that the product of the two middle terms of a G.P. consisting of an *even* number of terms is equal to the product of the first and last terms.

16. Sum to n terms

   (i) $1 + 4 + 13 + 40 + \cdots\cdots$ $\left[\frac{3}{4}(3^n - 1) - \frac{1}{2}n \right]$

   (ii) $3 + 5 + 9 + 17 + 33 + \cdots\cdots$ $\left[2^{n+1} + (n - 2) \right]$

17. Sum to n terms

   (i) $3 + 33 + 333 + 3333 + \cdots\cdots$ $\left[\frac{10(10^n - 1)}{27} + \frac{n}{3} \right]$

   (ii) $5 + 55 + 555 + 5555 + \cdots\cdots$ $\left[\frac{50}{81}(10^n - 1) - \frac{5}{9}n \right]$

18. Sum to n terms

$$\cdot7 + \cdot77 + \cdot777 + \cdots\cdots \qquad \left[\frac{7}{9}n - \frac{7}{81}\left(1 - \frac{1}{10^n}\right) \right]$$

19. Sum to n terms

   (i) $1 + 3x + 5x^2 + 7x^3 + \cdots\cdots,\ x \neq 1.$

$$\left[\frac{2x(1 - x^{n-1})}{(1 - x)^2} + \frac{1 - (2n - 1)x^n}{1 - x} \right]$$

   (ii) $3.2 + 5.2^2 + 7.2^3 + \cdots\cdots$ $\left[n.2^{n+2} - 2^{n+1} + 2 \right]$

   (iii) $1 + \frac{2}{3} + \frac{3}{3^2} + \frac{4}{3^3} + \cdots\cdots$ $\left[\frac{3^{n+1} - (3 + 2n)}{3.3^{n-2}} \right]$

20. Find the sum of the series in the 16th group of

$$1 + \left(\frac{1}{3} + \frac{1}{3^2}\right) + \left(\frac{1}{3^3} + \frac{1}{3^4} + \frac{1}{3^5}\right) + \cdots\cdots$$

$$\left[\frac{1}{2}\left\{ \left(\frac{1}{3}\right)^{119} - \left(\frac{1}{3}\right)^{135} \right\} \right]$$

21. Sum the following series to 12 terms

$$\frac{5}{2} + \frac{3}{2^2} + \frac{5}{2^3} + \frac{3}{2^4} + \cdots\cdots \qquad \left[4\frac{1361}{4096} \right]$$

$$\left[\text{Write the series as } \frac{10+3}{2^2} + \frac{10+3}{2^4} + \cdots\cdots \right]$$

22. Sum to n groups

   (i) $1 + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2} + \frac{1}{2^2}\right) + \left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3}\right) + \cdots$

$$\left[2n - 2 + \frac{1}{2^{n-1}} \right]$$

(ii) $(x + y) + (x^2 + xy + y^2) + (x^3 + x^2y + xy^2 + y^3) + \cdots$

$$\left[\frac{1}{x-y}\left\{x^2 \cdot \frac{1-x^n}{1-x} - y^2 \frac{1-y^n}{1-y}\right\}\right]$$

23. The sum of three numbers in G.P. is 38 and their product is 1728 ; find them.   [8, 12, 18 ; *or* 18, 12, 8]

24. If the sum of three numbers in G.P. be 7 and the sum of their squares be 21, find them.   [4, 2, 1]

25. The product of three numbers in G.P. is 27 and the sum of their products in pairs is 39 ; find the numbers.
          [1, 3, 9]

26. The arithmetic mean of two numbers is 15 and their geometric mean is 9. Find the numbers. [3 *and* 27]

27. Find the ratio of two numbers when their A.M. is to their G.M. as 5 : 3.   [9 : 1]

28. The A.M. of two numbers is to their G.M. as $m : n$. Show that the numbers are as

$$m + \sqrt{(m^2 - n^2)} : m - \sqrt{(m^2 - n^2)}.$$

29. Show that the square of the A.M. of two quantities is equal to the A.M. of the A.M. and G.M. of the squares of the same quantities.

30. If A be the A.M. and G be the G.M. between two numbers, show that the numbers are given by

$$A \pm \sqrt{\{(A + G)(A - G)\}}.$$

31. Insert

(i) 3 geometric means between 4 and 324.

          [± 12, 36, ± 108]

(ii) 6 geometric means between 56 and $-\frac{7}{16}$.

          [-28, 14, -7, $\frac{7}{2}$, $-\frac{7}{4}$, $\frac{7}{8}$]

32. Find the sum of the n geometric mean inserted between a and b.

$$\left[\left(ab^{\frac{1}{n+1}} - ba^{\frac{1}{n+1}}\right) \div \left(a^{\frac{1}{n+1}} - b^{\frac{1}{n+1}}\right)\right]$$

33. If one G.M. (G) and two A.M.'s (p, q) are inserted between two given quantities, prove that

$$G^2 = (2p - q)(2q - p).$$

34. If one A.M. (A) and two G.M.'s (p, q) are inserted between two given numbers, show that

$$\frac{p^2}{q} + \frac{q^2}{p} = 2A.$$

35. If a, b, c be in G.P. and x, y be A.M.'s between a, b and b, c respectively, show that

$$\frac{a}{x} + \frac{c}{y} = 2 \; ; \quad \frac{1}{x} + \frac{1}{y} = \frac{2}{b}.$$

36. Show that the arithmetic mean of the roots of $x^2 - 2ax + b^2 = 0$ is the geometric mean of the roots of $x^2 - 2bx + a^2 = 0$ and *vice versa*.

37. If $\dfrac{a + bx}{a - bx} = \dfrac{b + cx}{b - cx} = \dfrac{c + dx}{c - dx} \cdots\cdots\cdots$, then a, b, c, d are in G.P.

38. (i) Show that the pth, qth, rth terms of a G. P. are in G.P., if p, q, r be in A.P.

(ii) If the pth, qth, rth and sth terms of an A.P. be in G.P., then $p - q, q - r, r - s$ will be in G.P.

39. Show that the ratio of the sums of n terms of two geometrical series having the same common ratio is the ratio of their mth terms.

40. If S_n represent the sum of n terms of a given G.P., find the sum of

$$S_1 + S_2 + \cdots\cdots\cdots + S_n. \qquad \left[\frac{a}{r-1} \left\{ \frac{r(r^n - 1)}{r - 1} - n \right\} \right.$$

$$\textit{where } a = 1\textit{st term,}$$

$$\left. r = \textit{common ratio} \right]$$

18

16. Sum of an Infinite Geometric Series.[*]

Denoting the sum of n terms of the geometric series $a, ar, ar^2, \ldots\ldots$ by S_n, we have from Art. 12,

$$S_n = a \cdot \frac{1 - r^n}{1 - r} = \frac{a}{1 - r} - \frac{ar^n}{1 - r}.$$

Now, when r is a proper fraction positive or negative, the absolute value of r^n and consequently of $\frac{r^n}{1 - r}$ will decrease; as n increases; moreover, the value of $\frac{r^n}{1 - r}$ can be made as small as we please by sufficiently increasing the value of n.

Hence, *when r is numerically less than unity*, the value of S_n may be made to approach the value $\frac{a}{1 - r}$ as nearly as we please by making n large enough.

More shortly, when $-1 < r < +1$,

$$a + ar + ar^2 + \cdots + ar^{n-1} \to \frac{a}{1 - r}, \text{ as } n \to \infty$$

This fact is expressed by saying that the sum of an infinite number of terms (or the sum to infinity) of the series when r is numerically less than unity, is $\frac{a}{1 - r}$.

$$S = \frac{a}{1 - r}, \quad\quad \cdots \quad (5)$$

provided *r is numerically less than unity.*

---

[*] A series in which every term (however large its original number may be) is followed by another term, is called an *infinite* series. Although there is a last term for a finite series, an infinite series has no last term.

Note. The discussion of the infinite geometric series when the common ratio is not numerically less than unity is beyond the scope of this elementary treatise.

17. Recurring Decimal.

A recurring decimal is an example of an infinite geometric series. Thus,

(i) $\cdot 3 = \cdot 3333\cdots$ to $\infty = \dfrac{3}{10} + \dfrac{3}{10^2} + \dfrac{3}{10^3} + \cdots$ to ∞,

which is an infinite G.P., whose first term is $\frac{3}{10}$ and common ratio $\frac{1}{10}$.

$\therefore$ its sum $= \dfrac{3/10}{1 - 1/10} = \dfrac{3}{10} \times \dfrac{10}{9} = \dfrac{1}{3}$.

(ii) $\cdot 1\dot{2}\dot{4} = \cdot 124242424\cdots$ to $\infty = \dfrac{1}{10} + \left\{ \dfrac{24}{10^3} + \dfrac{24}{10^5} + \dfrac{24}{10^7} + \cdots \text{ to } \infty \right\}$

$= \dfrac{1}{10} +$ an infinite G.P., whose first term is $\dfrac{24}{10^3}$ and

common ratio $\dfrac{1}{10^2}$

$= \dfrac{1}{10} + \dfrac{24/10^3}{1 - 1/10^2} = \dfrac{1}{10} + \dfrac{24}{1000} \times \dfrac{100}{99} = \dfrac{1}{10} + \dfrac{24}{990} = \dfrac{99 + 24}{990}$

$= \dfrac{123}{990} = \dfrac{124 - 1}{990}$.

The reason for the Arithmetical *rule for converting a recurring decimal into a vulgar fraction* is now obvious.

Ex. *Convert* $\cdot 8\dot{3}\dot{6}$ *into a vulgar fraction.*

Let $S = \cdot 836363636\cdots$

$\therefore$ $10S = 8\cdot 36363636\cdots$

and $1000S = 836\cdot 36363636\cdots$

Hence, on subtraction $(1000 - 10)S = 836 - 8$.

$\therefore$ $S = \dfrac{836 - 8}{1000 - 10} = \dfrac{836 - 8}{990}$.

18. Illustrative Examples.

Ex. 1. *Sum to infinity :*

     $\frac{1}{2} - \frac{1}{4} + \frac{1}{8} - \frac{1}{16} + \cdots$ to ∞.

Here, first term (*i.e.*, a) $= \frac{1}{2}$; common ratio (*i.e.*, r) $= -\frac{1}{2}$ which is numerically less than unity,

$\therefore$ By formula (5), $S\infty = \dfrac{\frac{1}{2}}{1-(-\frac{1}{2})} = \dfrac{\frac{1}{2}}{\frac{3}{2}} = \dfrac{1 \times 2}{2 \times 3} = \dfrac{1}{3}$.

Ex. 2.   *Sum to infinity :*

$$1 + 2a + 3a^2 + 4a^3 + \cdots\cdots \text{ to } \infty \; (a^2 < 1).$$

[*It is an Arithmetico-Geometrical series*]

Let S denote the reqd. sum.

$\therefore$ $S = 1 + 2a + 3a^2 + 4a^3 + \cdots \text{ to } \infty$.

Now, multiplying both sides by the common ratio of the G.P. part

$$Sa = a + 2a^2 + 3a^3 + \cdots \text{ to } \infty$$

By subtraction,

$$S(1-a) = 1 + a + a^2 + a^3 + \cdots \text{ to } \infty$$
$$= \frac{1}{1-a} \text{ by formula (5).}$$

$\therefore$ $S = \dfrac{1}{(1-a)^2}$.

Ex. 3.   *Find the G.P., whose sum to infinity is* $3\frac{1}{8}$ *and whose second term is* $\frac{1}{2}$.

Let a be the first term and r the common ratio.

Then, by formula (5), $\dfrac{a}{1-r} = 3\frac{1}{8} = \frac{25}{8}$. $\cdots$ (1)

Also, $ar = \frac{1}{2}$.

From (1), $a = \frac{25}{8}(1-r)$; substituting this value of a in (2),

$\frac{25}{8}(1-r).r = \frac{1}{2}$, *i.e.*, $r^2 - r + \frac{4}{25} = 0$,

i.e., $(r-\frac{1}{2})^2 = \frac{1}{4} - \frac{4}{25} = \frac{25-16}{100} = \frac{9}{100}$.

$\therefore$ $r - \frac{1}{2} = \pm \frac{3}{10}$. $\therefore$ $r = \frac{1}{2} \pm \frac{3}{10} = \frac{4}{5} \text{ or } \frac{1}{5}$.

As both the values of r are less than unity, both are admissible.

Also, from (2), $a = \dfrac{1}{2r} = \dfrac{5}{2}$ where $r = \dfrac{1}{5}$ $\left.\vphantom{\dfrac{5}{2}}\right\}$
$= \dfrac{5}{8}$ where $r = \dfrac{4}{5}$

Hence, the series is

$$\frac{5}{2} + \frac{1}{2} + \frac{1}{10} + \cdots \text{ to } \infty \text{ ; or } \frac{5}{8} + \frac{1}{2} + + \frac{2}{5} + \cdots \text{ to } \infty.$$

Examples (BII)

Find the sum of the following series [*Ex. 1-13*] :—

1. $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots\cdots$ to ∞. $\qquad$ [2]

2. $9 - 6 + 4 - \cdots$ to ∞. $\qquad$ [$5\frac{2}{5}$]

3. $\frac{5}{8} + \frac{1}{2} + \frac{2}{5} + \frac{8}{25} + \cdots\cdots$ to ∞. $\qquad$ [$3\frac{1}{8}$]

4. $\sqrt{3} + \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}} + \cdots\cdots$ to ∞. $\qquad$ $\left[\dfrac{3\sqrt{3}}{2} \right]$

5. $1 + \frac{1}{\sqrt{2}} + \frac{1}{2} + \frac{1}{2\sqrt{2}} + \cdots\cdots$ to ∞. $\qquad$ [$2 + \sqrt{2}$]

6. $3 - 1 + \frac{1}{3} - \frac{1}{9} + \cdots\cdots$ to ∞. $\qquad$ [$2\frac{1}{4}$]

7. $(\sqrt{2}+1) + 1 + (\sqrt{2}-1) + \cdots\cdots$ to ∞. $\quad$ [$\frac{1}{2}(4 + 3\sqrt{2})$]

8. $(2+\sqrt{3}) + 1 + (2-\sqrt{3}) + \cdots\cdots$ to ∞. $\quad$ [$\frac{1}{2}(5 + 3\sqrt{3})$]

9. $2 - {\cdot}2 + {\cdot}02 - {\cdot}002 + \cdots\cdots$ to ∞. $\qquad$ [$1\frac{9}{11}$]

10. $\frac{3}{5} + \frac{4}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \frac{3}{5^5} + \frac{4}{5^6} + \cdots\cdots$ to ∞. $\qquad$ [$\frac{19}{24}$]

11. $\frac{2}{3} + \frac{5}{3^2} + \frac{2}{3^3} + \frac{5}{3^4} + \frac{2}{3^5} + \frac{5}{3^6} + \cdots\cdots$ to ∞. $\qquad$ [$1\frac{3}{8}$]

12. $\frac{a}{x} + \frac{b}{x^2} + \frac{a}{x^3} + \frac{b}{x^4} + \cdots\cdots$ to ∞. $\qquad$ $\left[\dfrac{ax+b}{x^2-1} \right]$

13. $1 + \frac{x}{1+x} + \frac{x^2}{(1+x)^2} + \cdots\cdots$ to ∞ [$x > 0$] $\quad$ [$1+x$]

14. Show that $\dfrac{1}{1+x^2} \left\{ 1 + \dfrac{2x}{1+x^2} + \left(\dfrac{2x}{1+x^2} \right)^2 + \cdots \text{ to } \infty \right\}$

$$= \frac{1}{(1-x)^2}.$$

15. The sum of an infinite G.P. is 2 and the sum of the first two terms is $\frac{3}{2}$. Find the series.

$$\left[1 + \tfrac{1}{2} + \tfrac{1}{4} + \cdots\cdots, \; 3 - \tfrac{3}{2} + \tfrac{3}{4} - \cdots\cdots \right]$$

16. The sum of an infinite G.P. is 2 and the 2nd term is $-\frac{3}{2}$; find the series. $\left[\; 3 - \frac{3}{2} + \frac{3}{4} - \cdots \;\right]$

17. The sum of an infinite G.P. is $5\frac{1}{3}$ and the sum of their squares is $4\frac{4}{63}$. Find the common ratio. $\left[\; \frac{3}{4} \;\right]$

18. Sum to infinity

$$1 + 3x + 5x^2 + 7x^3 + \cdots \cdots \text{ to } \infty \; (-1 < x < 1).$$
$$\left[\; (1+x) \div (1-x)^2 \;\right]$$

19. Find the sum of the series

$$1 - 5a + 9a^2 - 13a^3 + \cdots \cdots \text{ to } \infty \; (-1 < a < 1)$$
$$\left[\; (1-3a) \div (1+a)^2 \;\right]$$

20. Find by the method of summing an infinite geometric series the values of

(i) $\cdot\dot{2}$, (ii) $\cdot 1\dot{3}\dot{7}$. $\left[\text{ (i) } \frac{2}{9}, \text{ (ii) } \frac{69}{495} \right]$

21. Explain the apparent paradox that the sum of an infinite number of terms of a G.P. is a finite quantity.

22. In an infinite G.P. each term is equal to three times the sum of all the terms which follow it and the sum of the first two terms is 15. Find the sum to infinity of the series. $\left[\; 16 \;\right]$

23. The sum of an infinite number of terms of a G.P. is 2 and the sum of their cubes is $\frac{8}{19}$. Find the series.

$$\left[\; \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots \;\right]$$

24. Suppose a body moves eternally in this manner *viz.*, 20 miles in the first minute, 19 miles in the second, $18\frac{1}{20}$ miles in the third and so on. Find the outmost distance it can reach. $\left[\; 400 \text{ miles} \;\right]$

25. The diameter of a tree trunk at a certain date is 10 inches ; show that if it increases $\frac{1}{2}$ inch in the following year and if the increase in each year is $\frac{99}{100}$ of that in the preceding year, the tree will never be 5 ft. thick.

26. If s be the sum of an infinite number of terms of a G.P. and σ be the sum of the squares of the terms, then

$$a = \frac{2s\sigma}{s^2 + \sigma} \text{ and } r = \frac{s^2 - \sigma}{s^2 + \sigma}$$

where a is the first term and r the common ratio.

27. If s is the sum to infinity of a G.P., whose first term is a and S_n is the sum to n terms of the above series, then

$$S_n = s \left\{ 1 - \left(1 - \frac{a}{s} \right)^n \right\}.$$

28. If $y = x + x^2 + x^3 + \cdots\cdots$ to ∞ $(0 < x < \tfrac{1}{2})$,

then $x = y - y^2 + y^3 - \cdots\cdots$ to ∞.

SEC. C. HARMONIC PROGRESSIONS

19. Definition.

First Definition. A series of quantities is said to be in *Harmonical Progression* when their reciprocals are in arithmetical progression. Thus, corresponding to every harmonic series there are arithmetic series.

The quantities a, b, c, d,... are in H.P., if $\frac{1}{a}$, $\frac{1}{b}$, $\frac{1}{c}$, $\frac{1}{d}$, $\cdots$ are in A.P.

Second Definition. Three quantities are said to be in *Harmonical Progression*, if the *ratio of the first to the third* is equal to *the ratio of the difference between the first and the second to the difference between the second and the third.*

Thus, a, b, c are in H.P., if $\frac{a}{c} = \frac{a-b}{b-c}$.

Any number of quantities are said to be in *Harmonical Progression* when every three consecutive terms are in *Harmonical Progression*.

Note 1. The word 'Harmonical Progression' is abbreviated as *H.P.*

Note 2. That *the above two definitions are identical* can be seen as follows.

If a, b, c are in H.P., then $\dfrac{1}{a}$, $\dfrac{1}{b}$, $\dfrac{1}{c}$ are in A.P.

But if $\dfrac{1}{a}$, $\dfrac{1}{b}$, $\dfrac{1}{c}$ are in A.P., then $\dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b}$.

$\therefore \quad \dfrac{a-b}{ab} = \dfrac{b-c}{bc}. \qquad \qquad \therefore \quad \dfrac{a}{c} = \dfrac{a-b}{b-c}.$

Again, if a, b, c are in H.P., then $\dfrac{a}{c} = \dfrac{a-b}{b-c}$.

$\therefore \quad a\,(b-c) = c\,(a-b).$

Now, dividing every term by abc, we get

$$\frac{ab}{abc} - \frac{ac}{abc} = \frac{ac}{abc} - \frac{bc}{abc},$$

or, $\qquad \dfrac{1}{c} - \dfrac{1}{b} = \dfrac{1}{b} - \dfrac{1}{a}.$

$\therefore \qquad \dfrac{1}{a}$, $\dfrac{1}{b}$, $\dfrac{1}{c}$ are in A.P.

Note 3. It should be noted that no general formula (as in the case of A.P. or G.P.) can be found for the sum of any number of quantities in H.P.

20. Harmonic Mean.

If three quantities are in Harmonical Progression, the middle one is said to be the *Harmonic mean* between the other two.

Thus, if a, b, c are in H.P., then b is the Harmonic mean between a and c.

When a, b, c are in H.P., then $\dfrac{1}{a}$, $\dfrac{1}{b}$, $\dfrac{1}{c}$ are in A.P.

$\therefore \quad \dfrac{1}{b} - \dfrac{1}{a} = \dfrac{1}{c} - \dfrac{1}{b},$

$\qquad \dfrac{2}{b} = \dfrac{1}{a} + \dfrac{1}{c} = \dfrac{a+c}{ac}.$

$\therefore \quad b = \dfrac{2ac}{a+c}.$

To insert a given number of Harmonic means between two given quantities.

Let a and b be two given quantities, n being the number of means to be inserted. If m_1, m_2,...... m_n be the n harmonic means, then

$$\frac{1}{a}, \frac{1}{m_1}, \frac{1}{m_2}, \dotsc \frac{1}{m_n}, \frac{1}{b} \text{ are in A.P.}$$

of which $\frac{1}{b}$ is the $(n+2)$th term and $\frac{1}{a}$ is the first term.

If d be the c. d. of the series, then,

$$\frac{1}{b} = \frac{1}{a} + (n+1)\,d, \text{ whence } d = \frac{a-b}{(n+1)ab}.$$

Hence, the arithmetic means are

$$\frac{1}{a} + \frac{a-b}{(n+1)ab}, \frac{1}{a} + \frac{2(a-b)}{(n+1)ab}, \dotsc, \frac{1}{a} + \frac{n(a-b)}{(n+1)ab}.$$

The required harmonic means, being the reciprocals of these arithmetic means, are

$$\frac{(n+1)ab}{a+nb}, \frac{(n+1)ab}{2a+(n-1)b}, \dotsc \frac{(n+1)ab}{na+b}.$$

21. Relation among the Three means.

Let A, G, H denote the Arithmetic, Geometric and Harmonic means between a and b. Then, we have from Arts. 2, 9 and 20,

$$A = \frac{a+b}{2} \qquad \cdots \quad (1),$$

$$G = \sqrt{ab} \qquad \cdots \quad (2),$$

$$H = \frac{2ab}{a+b} \qquad \cdots \quad (3),$$

$$\therefore \ A.H. = \frac{a+b}{2} \cdot \frac{2ab}{a+b} = ab = G^2.$$

$\therefore$ G is the geometric mean between A and H.

Thus, *the geometric mean of any two quantities is also the geometric mean of their arithmetic and harmonic means.*

Again, $A - G = \dfrac{a+b}{2} - \sqrt{ab} = \tfrac{1}{2}(a + b - 2\sqrt{ab})$

$$= \tfrac{1}{2}(\sqrt{a} - \sqrt{b})^2$$

which is always positive, if a and b are positive and unequal.

$\therefore \quad A > G$, if a and b are positive and unequal.

Again since, $AH = G^2$ and $A > G$. $\quad \therefore \quad H < G$.

$\therefore \quad A > G > H$.

Thus, *the Arithmetic, Geometric and Harmonic means of any two positive and unequal quantities are in descending order of magnitude.*

22. Illustrative Examples,

Ex. 1. *The 4th term of an H.P.* $\tfrac{2}{3}$ *and 7th term is* $\tfrac{2}{5}$ *; find the nth term of the H.P.*

Let a be the first term and d the c.d. of the corresponding arithmetic series ; then since 4th and 7th terms of the A.P. are $\tfrac{3}{2}$ and $\tfrac{5}{2}$.

$\therefore \quad \begin{aligned} \tfrac{3}{2} &= a + 3d \cdots (1) \\ \tfrac{5}{2} &= a + 6d \cdots (2) \end{aligned} \Big\} \quad \therefore \quad \begin{aligned} a &= \tfrac{1}{2} \\ d &= \tfrac{1}{3}. \end{aligned}$

$\therefore \quad n$th term of the A.P. $= \tfrac{1}{2} + (n-1)\tfrac{1}{3} = \tfrac{1}{3}n + \tfrac{1}{6} = \dfrac{2n+1}{6}$.

$\therefore \quad n$th term of the H.P. $= \dfrac{6}{2n+1}$.

Ex. 2. *Find the 10th term of the series*

$$\tfrac{2}{5} + \tfrac{3}{20} + \tfrac{6}{15} + \tfrac{1}{5} + \cdots$$

Since, the reciprocal of the terms of the series *viz.*, $\tfrac{5}{2}$, $\tfrac{10}{3}$, $\tfrac{25}{6}$, $5 \ldots$, are in A.P. of which the 1st term is $\tfrac{5}{2}$ and c.d. $\tfrac{5}{6}$.

$\therefore \quad$ 10th term of A.P. $= \tfrac{5}{2} + 9 \cdot \tfrac{5}{6} = \tfrac{5}{2} + \tfrac{15}{2} = 10$.

$\therefore \quad$ 10th term of H.P. $= \tfrac{1}{10}$.

Ex. 3. *Insert 3 H.M.'s between 4 and 2.*

Let d denote the c.d. of the corresponding A.P.

Then, $\frac{1}{2}$ = 5th term of the A.P. of which $\frac{1}{4}$ is the first term.

$\therefore$ $\frac{1}{2} = \frac{1}{4} + 4.d.$ $\quad \therefore$ $4d = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$ $\quad \therefore$ $d = \frac{1}{16}.$

$\therefore$ The corresponding A.M.'s are $\frac{1}{4} + \frac{1}{16}, \frac{1}{4} + \frac{2}{16}, \frac{1}{4} + \frac{3}{16}$, *i.e.*, $\frac{5}{16}, \frac{6}{16}, \frac{7}{16}$.
Hence, the reqd. H.M.'s are $\frac{16}{5}, \frac{16}{6}, \frac{16}{7}$, *i.e.*, $3\frac{1}{5}, 2\frac{2}{3}, 2\frac{2}{7}$.

Ex. 4. *If a, b, c are in H.P., prove that*

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in H.P.}$$

Here, $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A.P.

$\therefore$ $\frac{a+b+c}{a}, \frac{a+b+c}{b}, \frac{a+b+c}{c}$ are in A.P. $\qquad$ [Multiplying all the terms by the same quantity $a+b+c$.]

$\therefore$ $\frac{a+b+c}{a} - 1, \frac{a+b+c}{b} - 1, \frac{a+b+c}{c} - 1$ are in A.P.,

i.e., $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A.P.

$\therefore$ $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ are in H.P.

Ex. 5. *If $a_1, a_2, \ldots\ldots a_n$ be in H.P., prove that*

$$a_1 a_2 + a_2 a_3 + \cdots + a_{n-1} a_n = (n-1) a_1 a_n.$$

Here, $\frac{1}{a_1}, \frac{1}{a_2}, \ldots \frac{1}{a_n}$ are in A.P.

Let d be the common difference.

Then, $\frac{1}{a_2} - \frac{1}{a_1} = d$, *i.e.*, $a_1 a_2 = \frac{a_1 - a_2}{d}$.

Similarly, $\frac{1}{a_3} - \frac{1}{a_2} = d$, *i.e.*, $a_2 a_3 = \frac{a_2 - a_3}{d}$,

$\frac{1}{a_4} - \frac{1}{a_3} = d$, *i.e.*, $a_3 a_4 = \frac{a_3 - a_4}{d}$,

$\therefore$ $\frac{1}{a_n} - \frac{1}{a_{n-1}} = d$ *i.e.*, $a_{n-1} a_n = \frac{a_{n-1} - a_n}{a}$.

Adding, $\Sigma a_1 a_2 = \frac{a_1 - a_n}{d}$. $\qquad \cdots \qquad \cdots \qquad \cdots$ (1)

Since, $\frac{1}{a_n}$ is the nth term of the A.P. $\therefore \frac{1}{a_n} = \frac{1}{a_1} + (n-1)\,d.$

$\therefore (n-1)\,d = \frac{1}{a_n} - \frac{1}{a_1} = \frac{a_1 - a_n}{a_1 a_n}.$ $\therefore \frac{a_1 - a_n}{d} = (n-1)\,a_1 a_n \cdots (2)$

From (1) and (2), the reqd. result follows.

Examples (C)

1. Find the 14th term of the series $\frac{6}{5}, 1, \frac{6}{7}, \ldots \ldots$ $\qquad [\,\frac{1}{3}\,]$

2. The 7th term of an H.P. is $\frac{1}{10}$ and the 12th term is $\frac{1}{25}$. What is the 20th term ?

$\qquad [\,\frac{1}{49}\,]$

3. The 3rd term of an H.P. is $\frac{3}{11}$ and the 7th term is 1. Find the series.

$\qquad [\,\frac{1}{5}, \frac{3}{13}, \frac{3}{11}, \frac{1}{3}, \ldots \ldots\,]$

4. Find the nth term of the series

$$4 + 4\tfrac{2}{7} + 4\tfrac{8}{13} + 5 + \cdots\cdots$$

$$\left[\,\frac{60}{16-n}\,\right]$$

5. If the mth term of an H.P. is n and the nth term is m, then rth term will be $\frac{mn}{r}$.

6. Insert

(i) 4 H.M. between 1 and 30. $\qquad [\,\frac{150}{121}, \frac{76}{46}, \frac{50}{21}, \frac{75}{17}\,]$

(ii) 6 H.M. between 3 and $\frac{6}{23}$. $\qquad [\,\frac{6}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}\,]$

7. There are n harmonic means between 1 and 4 such that the 1st mean : last mean $= 1 : 3$: find n. $\qquad [\,11\,]$

8. If x, y, z are in H.P., prove that

(i) $\dfrac{x}{y+z-x}, \dfrac{y}{z+x-y}, \dfrac{z}{x+y-z}$ are in H.P.

(ii) $\dfrac{x}{ax+b}, \dfrac{y}{ay+b}, \dfrac{z}{az+b}$ are in H.P.

(iii) $\dfrac{1}{x} + \dfrac{1}{y+z}, \dfrac{1}{y} + \dfrac{1}{z+x}, \dfrac{1}{z} + \dfrac{1}{x+y}$ are in H.P.

9. If a, b, c are in A.P., prove that

$$\frac{bc}{a(b+c)}, \frac{ca}{b(c+a)}, \frac{ab}{c(a+b)} \text{ will be in H.P.}$$

10. (i) If a, b, c in H.P., prove that

(a) bc, ca, ab are in A.P.   (b) $a(b+c)$, $b(c+a)$, $c(a+b)$ are in A.P.   (c) $(b+c-a)^2$, $(c+a-b)^2$, $(a+b-c)^2$ are in A.P.

(ii) If a^2, b^2, c^2 be in A.P., prove that

$b+c$, $c+a$, $a+b$ will be in H.P.

11. (i) Prove that a, b, c are in A. P., G. P. or H. P. according as

$$\frac{a-b}{b-c} = \frac{a}{a} \quad \text{or} \quad \frac{a}{b} \quad \text{or} \quad \frac{a}{c}.$$

(ii) Show that $\dfrac{a(b-c)}{a-b} = a$, b or c according as a, b, c are in A.P., G.P. or H.P.

12. Show that $b^2 > = < ac$ according as a, b, c are in A.P., G.P. or H.P.

13. If a, b, c are the pth, qth and rth terms of and H.P., show that

$$bc(q-r) + ca(r-p) + ab(p-q) = 0.$$

14. If $\dfrac{K-x}{ax} = \dfrac{K-y}{by} = \dfrac{K-z}{cz}$ and a, b, c be in A.P., show that x, y, z are in H.P.

15. If x_1, x_2, x_3, x_4 are in H.P., prove that

$$x_1 x_2 + x_2 x_3 + x_3 x_4 = 3 x_1 x_4. \qquad [\textit{Pat. 1921}]$$

16. If a, b, c be in H.P., show that

$$a : (a-b) = (a+c) : (a-c). \qquad [\textit{Pat. 1923}]$$

17. $\dfrac{x^2}{yz}$, $\dfrac{y^2}{zx}$, $\dfrac{z^2}{xy}$ are in H. P. ; x^3, y^3, z^3 are also in H.P.

18. If bc, ca, ab are in H.P.,

$$a\left(\frac{1}{b} + \frac{1}{c}\right), \ b\left(\frac{1}{c} + \frac{1}{a}\right), \ c\left(\frac{1}{a} + \frac{1}{b}\right) \text{ are in A.P.}$$

19. The A.M. of two numbers exceeds the G.M. by $\frac{3}{2}$ and the G.M. exceeds the H.M. by $\frac{6}{5}$. Find the numbers.

[12 and 3]

20. If a_1, a_2 be the A.M.'s, h_1, h_2 the H.M.'s and g_1, g_2 the G.M.'s between a and b, show that

$$a_1 h_2 = a_2 h_1 = g_1 g_2 = ab.$$

21. If H_1, H_2,......H_n be the harmonic means between a and b, show that

$$\frac{H_1 + a}{H_1 - a} + \frac{H_n + b}{H_n - b} = 2n.$$

22. If A be the arithmetic mean and H the harmonic mean between two quantities a and b, show that

$$\frac{a - A}{a - H} \times \frac{b - A}{b - H} = \frac{A}{H}.$$

23. If A.M. between a and b is equal to m times the H.M., then will $a : b = \sqrt{m} + \sqrt{(m - 1)} : \sqrt{m} - \sqrt{(m - 1)}$.

24. If G.M. between a and b is equal to m times the H.M., then will $a : b = m + \sqrt{(m^2 - 1)} : m - \sqrt{(m^2 - 1)}$.

25. If H is the H.M. between a and b, it is also the H.M. between x and y, where x is the H.M. between a and H and y the H.M. between b and H.

26. If three positive numbers a, b, c be in H.P., then

(i) $a^2 + c^2 > 2b^2$.   [*P. U. 1947*]

(ii) $a^3 + c^3 > 2b^3$.   [*P. U. 1938*]

(iii) $a^n + c^n > 2b^n$ (n being a positive integer).

27. If a, b, c be in A.P., l, m, n be in H.P. and al, bm, cn be in G.P., show that $a : b : c = 1/n : 1/m : 1/l$.

28. If $a^x = b^y = c^z$ and a, b, c are in G.P., then x, y, z are in H.P.

29. If x, y, z be in A.P. and y, z, x be in H.P., then z, x, y are in G.P.

30. If x is the A.M. between y and z, y be the G.M. between z and x, then will z be the H.M. between x and y.

31. If $(b+c)$, $(c+a)$, $(a+b)$ be in H. P., show that a^2, b^2, c^2 are in A.P.

32. If a, b, c, d be in A.P., then
$$bcd, cda, dab, abc \text{ are in H.P.}$$

33. If a, b, c, d are positive numbers. then
$$a+d = > \text{ or } < b+c$$
according as a, b, c, d are in A.P., G.P. or H.P.

34. If $\dfrac{b^2+c^2-a^2}{2bc}$, $\dfrac{c^2+a^2-b^2}{2ca}$, $\dfrac{a^2+b^2-c^2}{2ab}$ are in A.P., prove that $b+c-a, c+a-b, a+b-c$ are in H.P.

35. If $A_1, A_2, A_3, \ldots A_n$ are in A. P. and $H_1, H_2, \ldots H_n$ are in H. P., show that
$$A_p H_p (H_q - H_r) + A_q H_q (H_r - H_p) + A_r H_r (H_p - H_q) = 0$$
(where p, q, r are each less than n).

THE CONCEPT OF THE CONVERGENCE AND DIVERGENCE OF THE INFINITE SERIES

23. Definitions. An expression in which the successive terms are prepared according to some definite law is called a **Series**.

Ex. 1. The **Arithmetic** series :

(i) $a + (a+\beta) + (a+2\beta) + \cdots \{a + (n-1)\beta\} + \cdots$

(ii) $1 + 2 + 3 + \cdots + n + \cdots$

Ex. 2. The **Geometric** series :

(i) $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$

(ii) $1 + r + r^2 + \cdots r^{n-1} + \cdots$

Ex. 3. The **Harmonic** series :

(i) $\dfrac{1}{a} + \dfrac{1}{a+\beta} + \dfrac{1}{a+2\beta} + \cdots + \dfrac{1}{a+(n-1)\beta} + \cdots$

(iii) $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \dfrac{1}{n} + \cdots + \cdots$

Ex. 4. The **Arithmetico-Geometric** series :

(i) $1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots$

(ii) $1 + 3^2 x + 5^2 x^2 + \cdots + (2n-1)^2\, x^{n-1} + \cdots$

If a series terminates at some pre-assigned term, it is called a **finite series**.

Ex. 5. $1 + 3 + 5 + \cdots + (2n-1)$.

Ex. 6. $1 + 3 + 3^2 + \cdots 3^{n-1}$.

Ex. 7. $1 - \dfrac{1}{2} + \dfrac{1}{2^2} - \dfrac{1}{2^3} + \cdots + \left(-\dfrac{1}{2}\right)^{n-1}$.

Ex. 8. $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots + \dfrac{(-1)^{n-1}}{(2n-1)}.$

Ex. 9. $1 - \dfrac{1}{2} + \dfrac{1}{3} - \dfrac{1}{4} + \cdots + \dfrac{(-1)^{n-1}}{n}.$

If the number of terms of a series is unlimited, it is called an **infinite series.**

Ex. 10. $1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots$

Ex. 11. $1 + x + x^2 + \cdots + x^{n-1} + \cdots$

Ex. 12. $1 + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \cdots + \dfrac{1}{n^2} + \cdots$

Ex. 13. $1 + 3x + 5x^2 + \cdots + (2n-1)\, x^{n-1} + \cdots$

Note 1. In the above series, n the symbol for the number of terms is evidently a positive integer.

Note 2. We usually denote a series of positive terms by $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$; and a series of alternately positive and negative terms by $u_1 - u_2 + u_3 - u_4 + \cdots + (-1)^{n-1}\, u_n + \cdots$

24. For the Arithmetic series,
$$1 + 3 + 5 + \cdots + (2n-1) + \cdots,$$
the sum to n terms is n^2, which is a function of n.

For the Geometric series, $1 + x + x^2 + \cdots + x^{n-1} + \cdots$, the sum to n terms is $\dfrac{1-x^n}{1-x}$, which is also a function of n.

For the Arithmetico-Geometric series,
$$S = 1 + 2x + 3x^2 + \cdots + nx^{n-1} + \cdots,$$

S_n (the sum to n terms) $= 1 + 2x + 3x^2 + \cdots + nx^{n-1}$

$\therefore\ -xS_n \qquad\qquad = -x - 2x^2 - \cdots\cdots (n-1)\, x^{n-1} - nx^n$

Adding, $\therefore\ (1-x)\, Sn = (1 + x + x^2 + \cdots + x^{n-1}) - nx^n$
$$= \dfrac{1-x^n}{1-x} - nx^n.$$

$\therefore\quad S_n = \dfrac{1-x^n}{(1-x)^2} - \dfrac{nx^n}{(1-x)}$, which is also a function of n.

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But for the Harmonic series, $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots$.

$S_n \equiv 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} = $ the sum to n terms, cannot be expressed in a simpler form; and yet this sum S_n (which evidently depends on n) must be considered as a function of n.

Let the sum of n terms of the series, $u_1 + u_2 + \cdots + u_n + \cdots$, be $S_n \equiv u_1 + u_2 + \cdots + u_n$. Then, S_n shall also be a function of n.

Now let $n \to \infty$ (n tend to infinity). Then,

(i) S_n may $\to$ a finite and definite value S;

(ii) S_n may $\to \infty$;

(iii) S_n may $\to -\infty$;

(iv) S_n may not tend to a finite limit or to $+\infty$ or to $-\infty$. In this case as $n \to \infty$ through two different sets of values of n, $S_n \to$ two different limits.

In case (i), we say that the original series is **convergent**. In cases (ii) and (iii), it is called **divergent**; and in case (iv), the series is called **oscillatory**.

Ex. 14. Show that the *Geometric* series : $1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-1}} + \cdots$ is Convergent.

Here, $S_n = \dfrac{1 - \frac{1}{2^n}}{1 - \frac{1}{2}} = 2\left(1 - \frac{1}{2^n}\right) = 2 - \dfrac{1}{2^{n-1}}$, which $\to 2$, as $n \to \infty$, for instance,

$S_{11} = 2 - \dfrac{1}{2^{10}} = 2 - \dfrac{1}{1024}$; $S_{21} = 2 - \dfrac{1}{2^{20}} = 2 - \dfrac{1}{1048576}$; etc., which become nearer and nearer to 2, as n increases. $\therefore$ The series is **Convergent**.

Ex. 15. Show that the *Arithmetic* series, $1+2+3+\cdots+n+\cdots$ is **divergent**.

Here, $S_n = \frac{1}{2}n\,(n+1) = \frac{1}{2}\,(n^2+n)$ which $\to \infty$, as $n \to \infty$. $\therefore$ The series is **divergent**.

Ex. 16. Show that the *Geometric* series, $1-1+1-1+-\cdots$, is **oscillatory**.

Let $m = $ a positive integer. Then
$$S_{2m} = (1-1)+(1-1)+(1-1)+\cdots \text{ to } m \text{ groups}$$
$$= 0+0+0+\cdots \text{ to } m \text{ terms}$$
$$= 0. \qquad\qquad \cdots \qquad\qquad \cdots \quad (1)$$

But $S_{2m+1} = 1+(-1+1)+(-1+1)+(-1+1)+\cdots$ to m groups.
on setting aside the first term 1,
$$= 1+(0+0+0+\cdots \text{ to } m \text{ terms}) = 1. \qquad \cdots \quad (2)$$

Thus, for the given series, $S_n \to 0$, when $n = 2m$, an *even*, positive integer ; and $S_n \to 1$, when $n = 2m+1$, an *odd* positive integer. $\therefore$ The series is **oscillatory**.

25. Def. (*a*) An infinite series is said to be **convergent,** when the sum of the first n terms tends to a finite limit as n tends to infinity.

(*b*) An infinite series is said to be **divergent,** when the sum of the first n terms tends to $+\infty$ or $-\infty$ as n tends to infinity.

(*c*) An infinite series is said to be **oscillatory,** when the sum of the first n terms S_n does not tend to a finite limit or to $+\infty$ or to $-\infty$.

In this case as n tends to ∞ through different sets of values of n, $S_n \to$ different limits. If the limits are finite, the series is said to *oscillate finitely* and if the limits are infinite with different signs, the series is said to *oscillate infinitely* or *Oscillatory divergent*.

Ex. 17. Show that the *Geometric* series, $1 + 2 + 2^2 + \cdots + 2^{n-1} + \cdots$ is **divergent**.

Here, $S_n = \dfrac{2^n - 1}{2 - 1} = (2^n - 1)$, which $\to \infty$, as $n \to \infty$. $\therefore$ The given series is **divergent**.

Ex. 18. Show that the *Geometric* series, $1 - 2 + 2^2 - 2^3 + \cdots + (-2)^{n-2} + \cdots$ is **oscillatory divergent**.

Here, $S_{2m} = \dfrac{1 - (-2)^{2m}}{1 - (-2)} = \dfrac{1 - 2^{2m}}{3} \to -\infty$, as $m \to \infty$,

and $S_{2m+1} = \dfrac{1 - (-2)^{2m+1}}{1 - (-2)} = \dfrac{1 + 2^{2m+1}}{3} \to +\infty$, as $m \to \infty$.

$\therefore$ $\mathbf{S_1 \to -\infty, \ S_2 \to +\infty}$.

$\therefore$ The given series is **oscillatory divergent**.

26. The Geometric Series.

The Geometric series is

$$\Sigma \equiv a + ar + ar^2 + ar^3 + ar^{n-1} + \cdots$$
$$= a(1 + r + r^2 + r^3 + \cdots + r^{n-1} + \cdots). \quad \cdots \quad (1)$$

Hence, to test the Geometric series (1) for convergence, we need only consider the auxiliary series,

$$S \equiv 1 + r + r^2 + r^3 + \cdots + r^{n-1} + \cdots \qquad \cdots \quad (2),$$

for the series $\Sigma = aS$, when Σ and S are convergent.

Now, $S_n =$ sum to n terms of (2)

$$= 1 + r + r^2 + r^3 + \cdots + r^{n-1} = \dfrac{1 - r^n}{1 - r}.$$

When $-1 < r < 1$, the limit of r^n, as $n \to \infty$, shall $\to 0$. Hence, the limit of

$$S_n \ (\text{as } n \to \infty) \to \dfrac{1}{1 - r}; \ -1 < r < 1.$$

Hence, S, and therefore Σ, is **convergent** when $-1 < r < 1$.

In fact, $S \to \dfrac{1}{1-r}$ and $\Sigma \to a/(1-r)$, when $-1 < r < 1. \cdots$ (3)

When $r > 1$, $S_n = \dfrac{1-r^n}{1-r} = \dfrac{r^n-1}{r-1} \to +\infty$, as $n \to \infty$, for the

limit of $r^n \to \infty$, when $r > 1$ and $n \to \infty$. So the series S, and therefore Σ, is **divergent** when $r > 1$.

When $r < -1$, $S_n = \dfrac{1-r^n}{1-r} \to +\infty$ if n be *odd*, and $\to -\infty$,

when n is *even*, for $(1-r) > 0$ here, and $-r^{2m} \to -\infty$, $-r^{2m+1} \to +\infty$, where m is a positive integer and $r < -1$, (*e.g.*, $r = -2$). Hence, S, and therefore Σ, now **oscillatory divergent**, *i.e.*, Σ *oscillates infinitely.*

When $r = +1$, $S_n = 1+1+1\cdots$ to n terms $= n$, which $\to \infty$, as $n \to \infty$. Hence, S, and therefore Σ, is now **divergent.**

When $r = -1$. $S_n = 1-1+1-1+-\cdots$

$\therefore$ $S_{2m} = (1-1)+(1-1)+(1-1)+\cdots$ to m groups $= 0$; $m = $ a positive integer ; and

$S_{2m+1} = 1+(-1+1)+(-1+1)+(-1+1)+\cdots$ to m pairs of brackets

$= 1$; $m = $ a positive integer.

Thus, $S_n = 0$, when n is an *even* positive integer,

and $\quad S_n = 1$, when n is an *odd* positive integer.

So, S oscillates between 0 and 1.

$\therefore$ Σ oscillates between 0 and a.

Hence, S, and therefore Σ, **oscillates finitely.**

27.   Test for convergence of non-summable series.

When the sum to n terms of a series cannot be expressed as a simple function of n, we should apply methods, other than those discussed above, to investigate the convergence, or otherwise, of a given series. A few of these methods are discussed in the two following sections.

Section A—Series of positive terms

28.   Theorem I.   *An infinite series in which all the terms are positive is* **divergent** *if each term is greater than some finite quantity, however small.*

[**Note.** The case of series of all negative term follows easily from that of a series of all positive terms. For, let

$$S' = -u_1 - u_2 - u_3 - \cdots u_n - \cdots$$
$$S = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$$

Then, $S' = -S.$ $\therefore$ S' is **convergent** if S is **convergent**; S' is **divergent**; if S is **divergent**; and S' is **oscillatory** if S is **oscillatory**.]

Let each term of the given series be $> a > 0$. Then, S_n (the sum to n terms of the given series) $> na \to \infty$, as $n \to \infty$, however small a may be, provided $a > 0$. Hence, the given series is **divergent.**

29.   Two important principles.

I. *If we add a* **finite** *number of terms to a given series, (or remove a* **finite** *number of terms from the given series), the property of* **convergence, or otherwise,** *of the given series is not altered.*

[For the sum of these extra terms (added to or subtracted from the given series) is a finite quantity.]

II. *If a series of positive terms be convergent, then a new series obtained by changing the sign of some or all of them shall also be* **convergent.**

[The sum is evidently the greatest numerically when all the terms are of the same sign. Hence, the result follows.]

30. Theorem II. *An infinite series of positive terms is* **convergent** *if from and after some fixed term, the ratio of each term to the preceding is less than some positive quantity which is itself less than unity.* (**D' Alembert's Ratio Test**).

Let us denote the series starting from the fixed term by :

$$S \equiv u_1 + u_2 + u_3 + u_4 + \cdots$$

Here, $0 < \dfrac{u_2}{u_1} < r < 1,\ 0 < \dfrac{u_3}{u_2} < r < 1,\ 0 < \dfrac{u_4}{u_3} < r < 1,$

where r is the given positive number $< $ unity. Then,

$$S = u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \frac{u_4}{u_1} + \cdots \right)$$

$$= u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}.\frac{u_2}{u_1} + \frac{u_4}{u_3}.\frac{u_3}{u_2}.\frac{u_2}{u_1} + \cdots \right]$$

$$< u_1 \left[1 + r + r^2 + r^3 + \cdots \right],$$

which is *convergent*, for, $0 < r < 1$, here. Hence, the given series is *convergent*.

Ex. 19. Discuss the convergence of

$$S \equiv 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + (n+1)x^n + \cdots \text{ when } x > 0.$$

Here $\dfrac{u^{n+1}}{u_n} = \dfrac{n+1}{n}, \quad x \to x, \text{ as } n \to \infty.$

Hence, the series will be *convergent* when $0 < x < 1$. When $x = 1,\ S = 1 + 2 + 3 + 4 + \cdots + n + \cdots$, for which S_n (the sum to n terms

$= \frac{1}{2}n(n+1) \to \infty$, as $n \to \infty$. $\therefore$ The series is *divergent* when $x=1$.

Also, when $x > 1$, $S = 1 + 2x + 3x^2 + 4x^3 + \cdots + nx^{n-1} + \cdots$

$$> 1 + 2 + 3 + 4 + \cdots + n + \cdots,$$

which is a *divergent* series, as shown above.

$\therefore$ S is *divergent* when $x > 1$.

Note. Here, $\frac{u_{n+1}}{u_n} = \frac{n+1}{n} x$, which will be > 0 and < 1, when

$0 < \frac{n+1}{n} x < 1$, or $0 < x < \frac{n}{n+1}$.

Moreover, $(n+1) x < n$. $\therefore$ $x < n(1-x)$.

$\therefore$ $n > \frac{x}{1-x}$, when $0 < x < 1$.

Suppose, $x = \frac{9}{10}$. Then, $n > 9$. This means that $u_{10} = 10 \left(\frac{9}{10}\right)^9$ $= 9 \left(\frac{9}{10}\right)^8 = u_9$; and $0 < \frac{u_{11}}{u_{10}} < 1$, $0 < \frac{u_{12}}{u_{11}} < 1$; and so on.

So, the terms do not begin to decrease until after the 10th term.

31. Theorem III. *An infinite series of positive terms is* **divergent** *if from and after some fixed term, the ratio of each term to the preceding term is greater than or equal to unity.* (**D' Alembert's Ratio Test**).

Let the series starting from the fixed term be :

$$S = u_1 + u_2 + u_3 + \cdots + u_n + \cdots. \text{ Then,}$$

$$S = u_1 \left[1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \cdot \frac{u_2}{u_1} + \cdots \right]$$

$$> u_1 [1 + r + r^2 + \cdots], \text{ where } r \geqslant 1 ;$$

and this series is *divergent*. Hence, the given series is *divergent*.

32. Theorem IV. *If there are two infinite series of positive terms, and if the ratio of the corresponding terms in the two series is always finite, then the two series are* **both convergent** *or* **both divergent.** (**Comparison Test**)

[**Note.** A series of positive terms cannot **oscillate**, for S_n the sum to n terms continually increases as n increases as n increases.]

Let $S = u_1 + u_2 + u_3 + u_4 + \cdots$,

$\Sigma = v_1 + v_2 + v_3 + v_4 + \cdots$,

where $0 < a < \dfrac{u_1}{v_1}, \dfrac{u_2}{v_2}, \dfrac{u_3}{v_3}, \dfrac{u_4}{v_4}, \cdots < b.$ $\cdots$ (1)

Then, $0 < av_1 < u_1$; $0 < av_2 < u_2$; $0 < av_3 < u_3$; $0 < av_4 < u_4$; etc.

$\therefore \ 0 < a(v_1 + v_2 + v_3 + v_4 + \cdots) < (u_1 + u_2 + u_3 + u_4 + \cdots).$

$\therefore \ 0 < a\,\Sigma < S.$. $\cdots$ $\cdots$ $\cdots$ (2)

Again, from (1), $0 < u_1 < bv_1$; $0 < u_2 < bv_2$; $0 < u_3 < bv_3$; $0 < u_4 < bv_4$; etc.

$\therefore \ 0 < (u_1 + u_2 + u_3 + u_4 + \cdots).$

$< b\,(v_1 + v_2 + v_3 + v_4 + \cdots).$

$\therefore \ 0 < S < b\Sigma.$ $\cdots$ $\cdots$ $\cdots$ (3)

From (2) and (3), $\therefore \ 0 < a < \dfrac{S}{\Sigma} < b.$ $\cdots$ (4)

Hence, if S be convergent, Σ is convergent ; and if S be divergent, Σ is also divergent ; and *conversely*.

Section B—Series of alternately positive and negative terms

33.   Theorem V. (I) *An infinite series in which the terms are alternately positive and negative is* **convergent** *if each term is less than the preceding term, and if the nth term* $\to 0$, *as* $n \to \infty$.

(II) [*If the absolute value of the nth term* $\to l$, *a finite quantity, as* $n \to \infty$, *the series shall be* **oscillatory.**]

Part I. Let the series be : $u_1 - u_2 + u_3 - u_4 + \cdots$,

Taking $n = 2m =$ an *even* positive integer,

$$S_{2m} = (u_1 - u_2) + (u_3 - u_4) + (u_5 - u_6) + \cdots$$

to (m groups)

$$> 0 \quad \cdots \quad (1), \quad \because \quad u_1 > u_2 > u_3 > u_4 \cdots$$

Also, $S_{2m} = u_1 - [(u_2 - u_3) + (u_4 - u_5) + (u_6 - u_7) + \cdots$

to ($m - 1$) groups $] - u_{2m}$

$$< u_1 \quad \cdots \quad (2).$$

From (1), we see that S_{2m} increases continually as m increases. From (2), we see that $S_{2m} < u_1$ $\therefore$ The limit of $S_{2m} \to$ a definite number $L < u_1$. $\qquad \cdots \quad (3)$

Moreover, $S_{2m+1} = S_{2m} + (u_{2m+1}),$ $\qquad \cdots \quad (4)$

where $u_{2m+1} \to 0$, as $m \to \infty$, $\quad \therefore \quad S_{2m+1} \to S_{2m} \to L$, as $m \to \infty$. $\therefore$ The given series is **convergent**, and the sum is L.

Part II. From (3), $S_{2m} \to L$, as $m \to \infty$,

From (4), $S_{2m+1} \to L + l$, as $m \to \infty$,

for, in this case, $u_{2m+1} \to l$, as $m \to \infty$. Hence, the given series **oscillates** between (L) and ($L + l$).

Ex. 20. Discuss the convergence of $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + - \cdots$ $\quad (1)$.

Here, $u_{2m} = -\dfrac{1}{2m} \to 0$, and $u_{2m+1} = \dfrac{1}{2m+1} \to 0$, as $m \to \infty$.

Also, $u_1 = 1 > u_2 = \frac{1}{2} > u_3 = \frac{1}{3} > u_4 = \frac{1}{4}$, etc.

$\therefore$ The given series is **convergent**.

Note. We have, $\log_e (1 + x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + - \cdots$ $\quad (2)$,

which is **convergent** when $-1 < x < 1$, for the corresponding series of positive terms, $x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \cdots$ with $x > 0$, is **convergent** when $0 < x < 1$, as is obvious from the fact that, when

$0 < x < 1$, $x + \frac{1}{2}x^2 + \frac{1}{3}x^2 + \frac{1}{4}x^4 + \cdots$ $< x + x^2 + x^3 + x^4 + \cdots$ which is a G. P. of common ratio x, with $0 < x < 1$, and is **Convergent** for $0 < x < 1$.

Putting $x = 1$ in (2), we get

$$\log_e 2 = 1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + - \cdots, \qquad \cdots \qquad \cdots \quad (3),$$

where the series on the right is convergent by this Ex. 20. Hence, (3) is a real equality. Thus, the series (2) is **convergent** for $-1 < x \leq 1$.

Ex. 21. Discuss the convergence of $\frac{2}{1} - \frac{3}{2} + \frac{4}{3} - \frac{5}{4} + - \cdots$

Here, $u_1 = \frac{2}{1} > u_2 = \frac{3}{2} > u_3 = \frac{4}{3} > u_4 = \frac{5}{4} > \cdots$

But $u_{2m+1} = \dfrac{2m+2}{2m+1} \to 1$, as $m \to \infty$. $\qquad \cdots \quad (4)$

Moreover, $S_{2m} = (\frac{2}{1} - \frac{3}{2}) + (\frac{4}{3} - \frac{5}{4}) + (\frac{6}{5} - \frac{7}{6}) + \cdots$ to m groups

$\qquad\qquad = (1 - \frac{1}{2}) + (\frac{1}{3} - \frac{1}{4}) + (\frac{1}{5} - \frac{1}{6}) + \cdots$ to m groups

$\to \log_e 2$, as $m \to \infty$, by the note to Ex. 20, above. $\qquad \cdots \quad (5).$

Hence, the given series **oscillates** between $(\log_e 2)$ and $(1 + \log_e 2)$.

34. The discussion of the convergence of the Pseudo-Harmonic series, $1 + \dfrac{1}{2^p} + \dfrac{1}{3^p} + \dfrac{1}{4^p} + \cdots + \dfrac{1}{n^p} + \cdots$, is beyond our scope ; and it forms the basis of the Tests for convergence in Higher Algebra.

APPENDIX II

THE GRAPHS OF $|x|$, x^n (n a positive integer).
$1/x$, $1/x^2$, $\log_a x$ and a^x

1. Graphs of x^n. (*All the graphs are drawn for points near the origin.*)

We shall here consider three different cases :

(A) $n =$ an *odd* positive integer ;

(B) $n =$ an *even* positive integer, including the graph of $y = |x|$ (absolute value of x) and the case of $n - 2p/(2q + 1)$, where p is a finite positive integer and q is a very large positive integer.

(C) $n = -1$, $n = -2$.

Case (A) *Graphs of* $y = x^n$, *where* $n = 1, 3, 5, (2m + 1)$, *where* **m** *is a large positive integer.*

For the graph of $y = x$, we tabulate the values of x and y as :

x	-4	-3	-2	-1	0	1	2	3	4
y	-4	-3	-2	-1	0	1	2	$3.$	4

For the graph of $y = x^3$, we exhibit the following tabulation :

x	-1.6	-1.4	-1.2	-1	-0.8	-0.6	-0.4
y	-4.1	-2.7	-1.7	-1	-0.5	-0.2	-0.06

0	0.4	0.6	0.8	1	1.2	1.4	1.6
0	0.06	0.2	0.5	1	1.7	2.7	4.1

For the graph of $y = x^5$, we prepare our table as follows :

x	$-1\cdot4$	$-1\cdot3$	$-1\cdot2$	-1	$-0\cdot8$	$-0\cdot6$	$-0\cdot4$
y	$-5\cdot4$	$-3\cdot7$	$-2\cdot5$	-1	$-0\cdot3$	$-0\cdot08$	$-0\cdot01$

0	$0\cdot4$	$0\cdot6$	$0\cdot8$	1	$1\cdot2$	$1\cdot3$	$1\cdot4$
0	$0\cdot01$	$0\cdot08$	$0\cdot3$	1	$2\cdot5$	$3\cdot7$	$5\cdot4$

For the graph of $y = x^{2m+1}$, where m is a large positive integer, we shall not prepare any table, but shall take recourse to *the method of limits.*

We exhibit the graphs for the above equations in the following figure (Fig. 5), where the scale is $0''5$ to the *unit*.

Fig. 5. Graphs of $y = x$, $y = x^3$, $y = x^5$. (Scale unit = $0''\cdot5$)

It is clear from the figure that $y = x$ represents the straight line BA. Moreover, each of the curves, $y = x^3$, $y = x^5$, $y = x^{2m+1}$ ($m =$ a positive integer), goes through each of the 3 points, $B\,(-1, -1)$, $O\,(0, 0)$ and $A\,(1, 1)$. The figure also shows that each of the curves, $y = x^3$, $y = x^5$, $y = x^{2m+1}$ ($m =$ positive integer $\geqslant 1$), touches the x-axis at the origin and also crosses it there ; (such a point is called a **point of inflexion** on the curve).

The above curves lie only in the 1st and 3rd quadrants, the portions in the 3rd quadrant being obtained from those in the 1st quadrant by taking their images first in the x-axis and then is the y-axis, (or in the reverse order).

As regards the graph of $y = x^{2m+1}$; where m is a large positive integer, we note that as m increases from 0 through 1, 2, etc., the part of the curve in $(-1 < x < 1)$, approaches
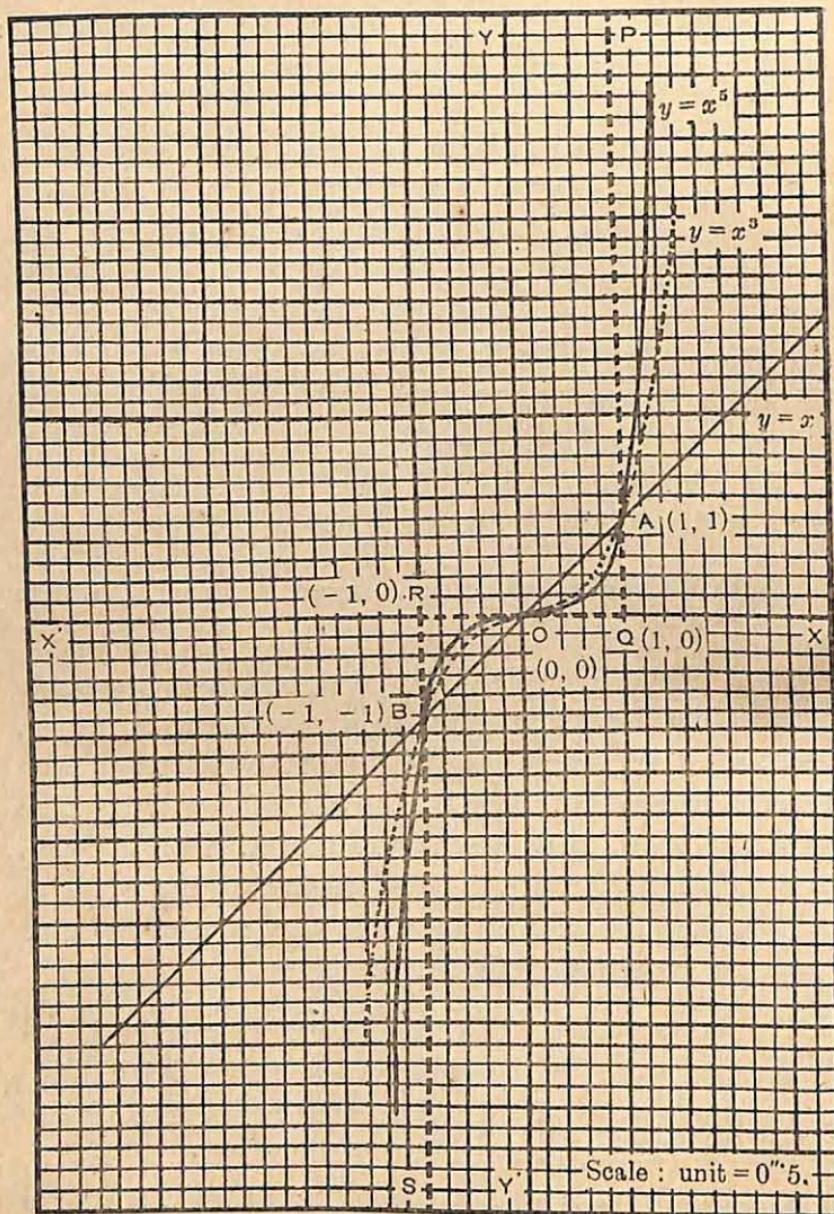
Fig. 5

more and more closely to the part RQ of the x-axis, the part BR of the line $x = -1$ and the part QA of the line $x = 1$; also, the parts of the curve in $(-\infty < x < -1)$ and $(1 < x < \infty)$ does so more and more closely to the part SB of the line $x = -1$ and the part AP of the line $x = 1$, respectively.

Hence, when m is very large (*i.e.*, $m \to \infty$, $m =$ integer), the curve, $y = x^{2m+1}$ cannot be differentiated from the broken line $SBROQAP$.

Case (B) *Graphs of* $y = x^2$, $y = x^4$, $y = x^{2m}$ (**m** = *a large positive integer*).

The graph of $y = |x| = $ *the absolute value of* **x** *is given for it has a relation to the graphs of* $y = x^{2m}$ (**m** = *a positive integer*) *very similar to what* $y = x$ *has in respect of the graphs of* $y = x^{2m+1}$ (**m** = *a positive integer*).

A note on the graphs of $y = x^{2p/(2q+1)}$, *where* **p, q** *are positive integers, is given in order to show what the graph of* $y = x^n$ *shall become as* **n** $\to 0$, *where* **n** = *a positive proper fraction with an* **even** *numerator and an* **odd** *denominator.*

The tabulation for $y = |x|$ is as follows :

x	-4	-3	-2	-1	0	1	2	3	4
y	4	3	2	1	0	1	2	3	4

That for $y = x^2$ is as :

x	-2	$-1\cdot8$	$-1\cdot6$	$-1\cdot4$	$-1\cdot2$	-1	$-0\cdot8$	$-0\cdot6$	$-0\cdot4$
y	4	$3\cdot2$	$2\cdot6$	$2\cdot0$	$1\cdot4$	1	$0\cdot6$	$0\cdot4$	$0\cdot2$

$-0\cdot2$	0	$0\cdot2$	$0\cdot4$	$0\cdot6$	$0\cdot8$	1	$1\cdot2$	$1\cdot4$	$1\cdot6$	$1\cdot8$	2
$0\cdot04$	0	$0\cdot04$	$0\cdot2$	$0\cdot4$	$0\cdot6$	1	$1\cdot4$	$2\cdot0$	$2\cdot6$	$3\cdot2$	4

That for $y = x^4$ is shown below as :

x	$-1\cdot4$	$-1\cdot2$	-1	$-0\cdot8$	$-0\cdot6$	$-0\cdot4$	0	$0\cdot4$
y	$3\cdot8$	$2\cdot1$	1	$0\cdot4$	$0\cdot1$	$0\cdot03$	0	$0\cdot03$

$0\cdot6$	$0\cdot8$	1	$1\cdot2$	$1\cdot4$
$0\cdot1$	$0\cdot4$	1	$2\cdot1$	$3\cdot8$

For the graph of $y = x^{2m}$, where m is a large positive integer, we shall not prepare any table, but shall take recourse to the *method of limits*.

For the gragh of $y = x^{2p/(2q+1)}$, where p, q are positive integers, we shall add a **note** in the proper place.

We exhibit the graphs for the above equations in the following figure (Fig. 6), where the scale is $0'' \cdot 5$ to the *unit*.

Fig. 6.   **Graphs of $y = |\,x\,|$, $y = x^2$, $y = x^4$. (Scale : unit $= 0''\cdot5$)**

It is clear from the figure that $y = |\,x\,|$ represents the broken line BOA lying in the first and second quadrants. Moreover, each of the curves, $y = x^2$, $y = x^4$, $y = x^{2m}$, ($m =$ a positive integer), goes through each of the 3 points. $B\,(-1, 1)$, $O\,(0, 0)$ and $A\,(1, 1)$. The figure also shows that each of the curves, $y = x^2$, $y = x^4$, $y = x^{2m}$, ($m =$ a positive integer $\geqslant 1$), touches the x-axis at the origin. ·

The above curves, is wholly above the x-axis in the *first* and *second* quadrants, the portions in the second quadrant being obtained from those in the first quadrant by a single image in the positive part of the y-axis.

As regards the gragh of $y = x^{2m}$, where m is a large positive integer, we note that as m increases through 1, 2,

etc., the part of the curve in $(-1 < x < 1)$ approaches more and more closely to the part BR of the line $x = -1$, the
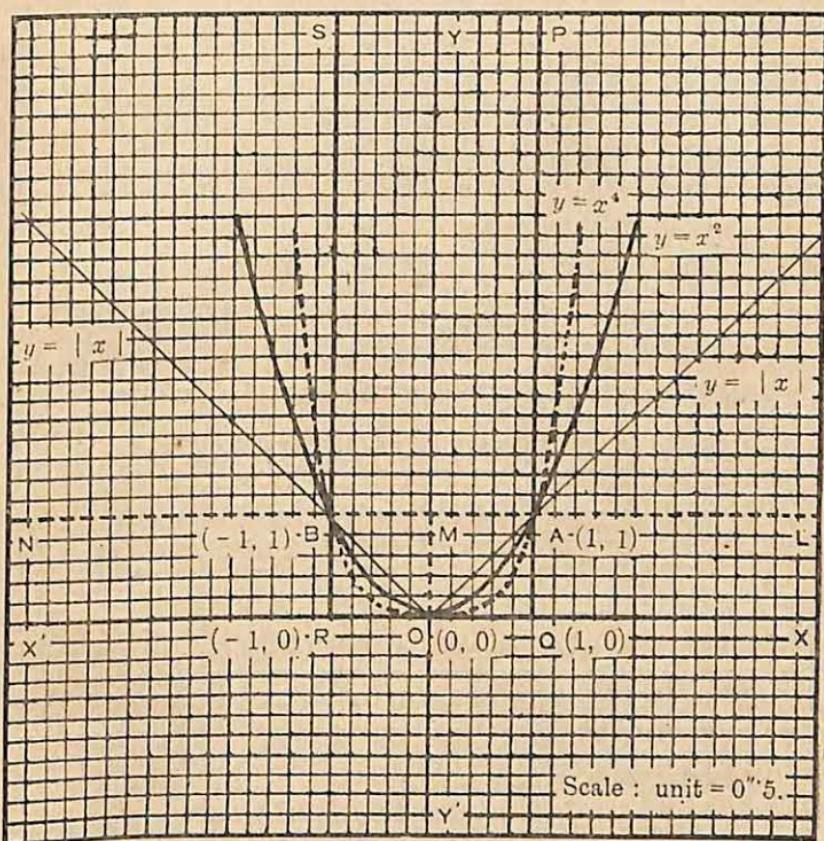
Fig. 6

part RQ of the x-axis and the part QA of the line $x = 1$. Also, the parts of the curve in $(-\infty < x < -1)$ and $(1 < x < \infty)$ approach more and more closely to the part SB of the line $x = -1$, and the part AP of the line $x = 1$, respectively. Hence, when m is very large (*i.e.*, $m \to \infty$, $m =$ integer), the curve, $y = x^{2m}$, cannot be differentiated from the broken line $SBROQAP$.

Note. The reader would do well to draw the graphs of $y = x^{2/3}$, $y = x^{2/5}$. Then, the graph of $y = x^{2p/(2q+1)}$, where p, q are positive integers and q is very large, would be found to be not distinguishable from the broken lines (OML, OMN).

When $n = 0$ (absolutely), the graph of $y = x^n$ is the same as that of $y = x^0$ or of $y = 1$, provided we take $y = 1$, when $x = 0$, to be a point on the graph; and is the line NML.

Case (C) *Graphs of* $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^2}$.

For the graphs of $y = \dfrac{1}{x}$, we add the following table :

x	-4	-3	-2.5	-2	-1.7	-1.25	-1
y	-0.25	-0.33	-0.4	-0.5	-0.6	-0.8	-1

	-0.8	-0.6	-0.5	-0.4	-0.33	-0.25
	-1.25	-1.7	-2	-2.5	-3	-4

x	0.25	0.33	0.4	0.5	0.6	0.8	1	1.25	1.7	2	2.5	3	4
y	4	3	2.5	2	1.7	1.25	1	0.8	0.6	0.5	0.4	0.33	0.25

For the graph of $y = \dfrac{1}{x^2}$, we have :

x	-4	-3	-2.5	-2	-1.67	-1.25	-1
y	0.06	0.1	0.2	0.3	0.4	0.6	1

-0.8	-0.6	-0.5	0.5	0.6	0.8	1	1.25
1.6	2.8	4	4	2.8	1.6	1	0.6

1.67	2	2.5	3	4
0.4	0.3	0.2	0.1	0.06

We exhibit the graphs for the above equations in the following figure (Fig. 7), where the scale is $0''\cdot5$ to the *unit*.
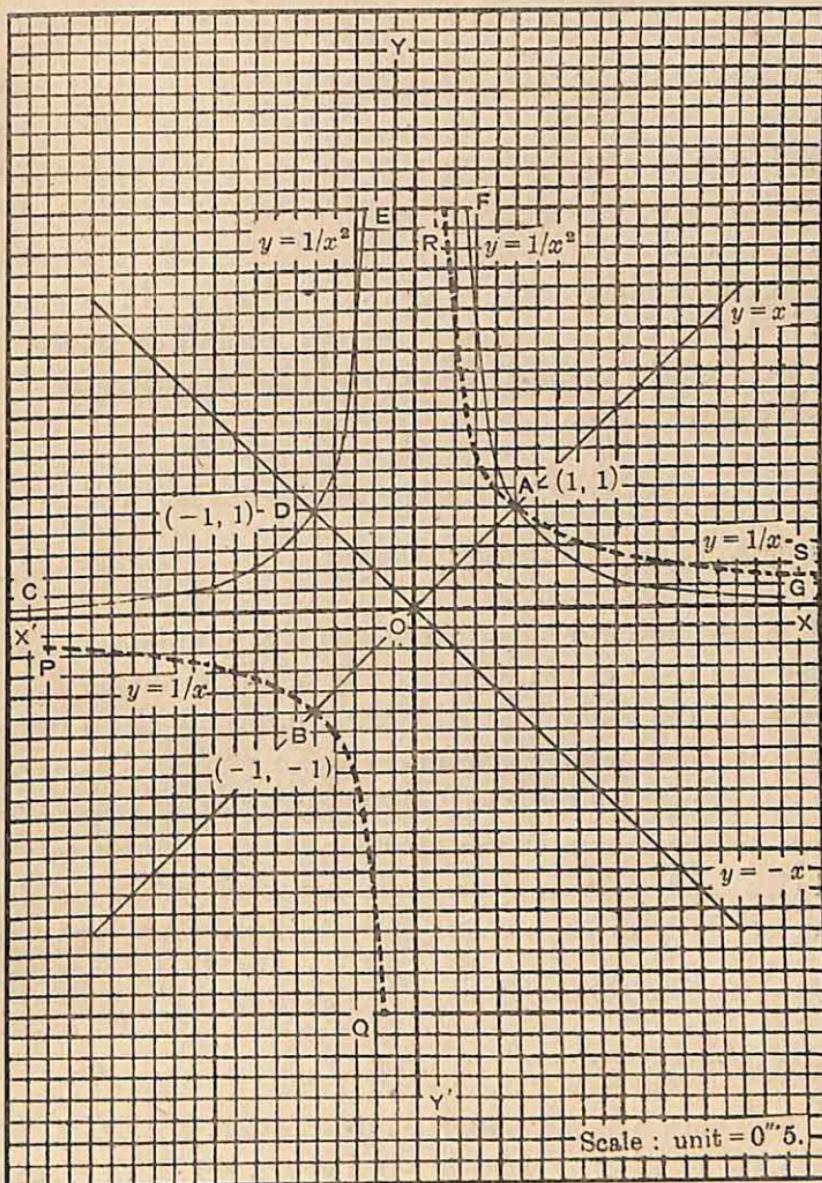
Fig. 7

Fig. 7.   Graphs of $y = \dfrac{1}{x}$, $y = \dfrac{1}{x^2}$.   **(Scale : unit = 0".5)**

It is well-known that the graph of $y = \dfrac{1}{x}$, $(i.e.,$ of $xy = 1)$, is a *rectangular* (or *equilateral*) hyperbola to which the x-axis and also the y-axis are **asymptotes**. This curve lies in the *first* and *third* quadrants, the part PBQ in the third quadrant being the image of the part RAS in the first quadrant with respect to the line $y = -x$. It is also symmetrical about the line $y = x$. The points $A\,(1,\,1)$ and $B\,(-1,\,-1)$ are the vertices of the hyperbola, and are nearest to the origin. There is a **discontinuity** in the graph as $x \to 0$, (or, $y \to 0$).

The curves, $y = \dfrac{1}{x}$ and $y = \dfrac{1}{x^2}$ cut one another at $A\,(1,\,1)$.

The graph of $y = \dfrac{1}{x^2}$ lies in the *first* and *second* quadrants.

It is **symmetrical** about the y-axis $(i.e.,$ the line, $x = 0)$. The part CDE of the curve in the second quadrant is the image of the part GAF in the first quadrant taken with respect to the positive side of the y-axis. The x-axis, is an asymptote to this curve at both sides ; but only the positive side of the y-axis is the second asymptote. The points $A\,(1,\,1)$ and $D\,(-1,\,1)$ are two important points on the curve. It is to be noted that points on the graphs of $y = \dfrac{1}{x^2}$ become **non-existent** as $x \to 0$, (or $y \to 0$).

2.   *Graphs of* $y = log_e\, x$, $y = log_{10}x$; $y = e^x$, $y = 10^x$.

The graphs of the **Logarithmic** function, $y = log_a\, x$, for different values of a, where $a > 0$, $a \neq 0$, $a \neq 1$, $a \neq \infty$, each consists of a single infinite branch.

The graphs of the **Exponential** function, $y = a^x$, for different values of a where $a > 0$, $a \neq 0$, $a \neq 1$, $a \neq \infty$, each consists of a single infinite branch.

It is to be noted that $a^{(\log_a x)} = x$ and $\log_a (a^x) = x$, so that the logarithmic and the exponential functions are **inverse** to one another.

For simplicity, we shall draw the graphs only of $y = \log_e x$, $y = \log_{10} x$; $y = e^x$, $y = 10^x$; and that also for points not far from the origin, (*i.e.*, in the **neighbourhood** or vicinity of the origin).

The necessary tabulations are exhibited below :—

(*i*) For the graph of $y = \log_e x$; $e = 2{\cdot}71828...$,

x	0·05	0·08	0·14	0·22	0·37 → 1/e	0·61	1	1·65
y	−3	−2·5	−2	−1·5	−1	−0·5	0	0·5

2·23	2·72 → e	3·32	3·67	4·06	4·48
0·8	1	1·2	1·3	1·4	1·5

(*ii*) For the graph of $y = \log_{10} x$.

x	0·01	0·05	0·1	0·25	0·5	1	2	3	4	5
y	−2	−1·3	−1	−0·6	−0·3	0	0·3	0·48	0·6	0·7

(*iii*) For the gragh of $y = e^x$; $e = 2{\cdot}71828...$,

x	−3	−2·5	−2	−1·5	−1	−0·5	0	0·5
y	0·05	0·08	0·14	0·22	0·37 → 1/e	0·61	1	1·65

0·8	1	1·2	1·3	1·4	1·5
2·23	2·72 → e	3·32	3·67	4·06	4·48

(*iv*) For the graph of $y = 10^x$.

x	-2	-1.3	-1	-0.6	-0.3	0	0.3	0.48	0.6	0.7
y	0.01	0.05	0.1	0.25	0.5	1	2	3	4	5

We exhibit the graphs for the above equations in the following figure (Fig. 8), where the scale is $0''.5$ to the *unit*.

**Fig. 8.   Graphs of $y = \log_e x$, $y = \log_{10} x$; $y = e^x$, $y = 10^x$.**

(Scale : unit = $0''.5$)

It is clear from the figure that the graph of $y = e^x$ is the image of that of $y = \log_e x$ in the line $y = x$. Also the graph of $y = 10^x$ is the image of that of $y = \log_{10} x$ with respect to the same line $y = x$.

The negative side of the y-axis is an asymptote to each of $y = \log_e x$ and $y = \log_{10} x$. The negative side of the x-axis is, similarly, an asymptote of each of $y = e^x$ and $y = 10^x$.

Moreover, as $x \to \infty$, y also $\to \infty$, for each of the above four curves.

The common point of $y = \log_e x$ and $y = \log_{10} x$ is the point A $(1, 0)$; and that for the curves $y = e^x$ and $y = 10^x$ is the point B $(0, 1)$.

The graphs of $y = \log_e x$ and $y = \log_{10} x$ are wholly to the *right* of the y-axis ; while those of $y = e^x$ and $y = 10^x$ are wholly *above* the x-axis.

For the curve, $y = e^x$, the slope at B $(0, 1)$ is $45°$, the curve being steeper to the *right* of B and more level to the *left*. For the curve $y = \log_e x$, the slope at $A(1, 0)$ is also $45°$, the curve being steeper to the *left* of A and more level to the right.
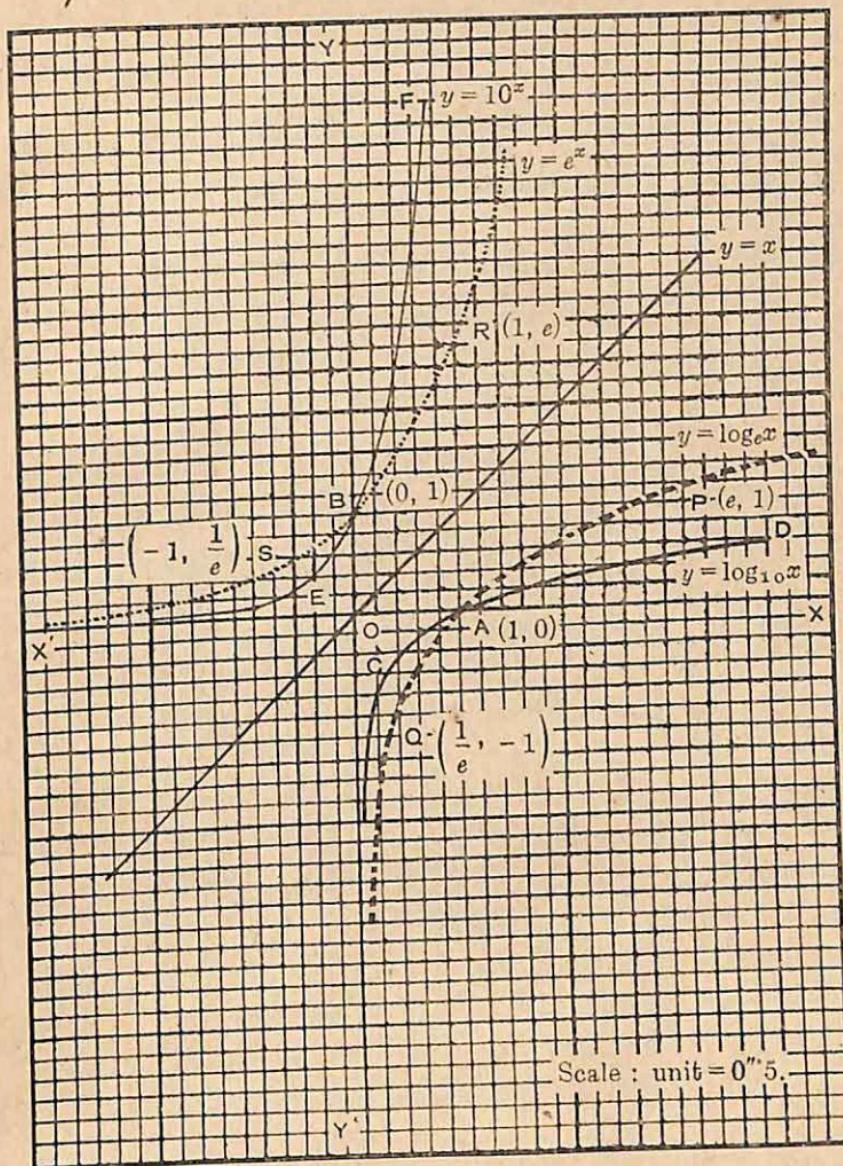
Fig. 8

Here, QAP is the graph of $y = \log_e x$, CAD is that of $y = \log_{10} x$; SBR is that of $y = e^x$ and EBF that of $y = 10^x$.

FIVE-FIGURE LOGARITHMIC TABLE

Explanation of the Table.

The table gives the common logarithms of all numbers from 1 to 10000, *i.e.*, those which consist of 4 digits or less. The tabulated quantities are the mantissæ only, correct to five places, with the decimal point dropped. The characteristic is to be supplied according to the rule given in Art. 98. The main body of the table gives logarithms (mantissa part) of numbers of 3 digits and the mean-difference table at the side supplies the increment in the mantissa due to the fourth digit. This increment, is written, in order to save space, giving the significant digits only, which are to be supplied with the necessary number of zeros to make up 5 places (here the table being a five-figure table). Thus ˙00024 will be written as 24 only in the difference table. As an example, to find log 2˙697, we notice from the table, that the mantissa for log 269 is ˙42975, and along the same row, the difference table gives 115 under the heading 7. This means that for 7 in the fourth place of the number (*i.e.*, for the number or 2697) the increment in the mantissa will be ˙00115. Hence log 2697 will have its mantissa ˙42975 + ˙00115 = ˙43090. Again, log 2˙697 has the same mantissa, but its characteristic is 0. Thus log 2˙697 = 0˙43090.

LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences								
											1	2	3	4	5	6	7	8	9
10	00000	00432	00860	01284	01703	02119	02531	02938	03342	03743	42	83	125	166	208	248	290	331	373
11	04139	04532	04922	05308	05690	06070	06446	06819	07188	07555	38	76	114	152	190	227	265	302	340
12	07918	08279	08636	08991	09342	09691	10037	10380	10721	11059	35	70	105	140	175	209	243	278	313
13	11394	11727	12057	12385	12710	13033	13354	13672	13998	14301	32	65	97	129	162	193	225	258	290
14	14613	14922	15229	15534	15836	16137	16435	16732	17026	17319	30	60	90	120	150	180	210	240	270
15	17609	17898	18184	18469	18752	19033	19312	19590	19866	20140	28	56	84	112	140	168	196	224	252
16	20412	20683	20952	21219	21484	21748	22011	22272	22531	22789	26	53	79	105	132	158	184	210	237
17	23045	23300	23553	23805	24055	24304	24551	24797	25042	25285	25	50	74	99	124	149	174	199	223
18	25527	25768	26007	26245	26482	26717	26951	27184	27416	27646	23	47	70	94	117	141	164	188	211
19	27875	28103	28330	28556	28780	29003	29226	29447	29667	29885	22	45	67	89	111	134	156	178	201
20	30103	30320	30535	30750	30963	31175	31387	31597	31806	32015	21	42	64	85	106	127	148	170	191
21	32222	32428	32634	32838	33041	33244	33445	33646	33846	34044	20	40	61	81	101	121	141	162	182
22	34242	34439	34635	34830	35025	35218	35411	35603	35793	35984	19	39	58	77	97	116	135	154	174
23	36173	36361	36549	36736	36922	37107	37291	37475	37658	37840	19	37	56	74	93	111	130	148	167
24	38021	38202	38382	38561	38739	38917	39094	39270	39445	39620	18	36	53	71	89	107	124	142	160
25	39794	39967	40140	40312	40483	40654	40824	40993	41162	41330	17	34	51	68	85	102	119	136	153
26	41497	41664	41830	41996	42160	42325	42488	42651	42813	42975	16	33	49	66	82	98	115	131	148
27	43136	43297	43457	43616	43775	43933	44091	44248	44404	44560	16	32	47	63	79	95	111	126	142
28	44716	44871	45025	45179	45332	45484	45637	45788	45939	46090	15	30	46	61	76	91	106	122	137
29	46240	46389	46538	46687	46835	46982	47129	47276	47422	47567	15	29	44	59	74	88	103	118	132

	0	1	2	3	4	5	6	7	8	9		1	2	3	4	5	6	7	8	9
30	47712	47857	48001	48144	48287	48430	48572	48714	48855	48996		14	29	43	57	72	86	100	114	129
31	49136	49276	49415	49554	49693	49831	49969	50106	50243	50379		14	28	42	55	69	83	97	110	125
32	50515	50651	50786	50920	51055	51188	51322	51455	51587	51720		13	27	40	54	67	80	94	107	121
33	51851	51983	52114	52244	52375	52504	52634	52763	52892	53020		13	26	39	52	65	78	91	104	117
34	53148	53275	53403	53529	53656	53782	53908	54033	54158	54283		13	25	38	50	63	76	88	101	113
35	54407	54531	54654	54777	54900	55023	55145	55267	55388	55509		12	24	37	49	61	73	86	98	110
36	55630	55751	55871	55991	56110	56229	56348	56467	56585	56703		12	24	36	48	60	71	83	95	107
37	56820	56937	57054	57171	57287	57403	57519	57634	57749	57864		12	23	35	46	58	70	81	93	104
38	57978	58092	58206	58320	58433	58546	58659	58771	58883	58995		11	23	34	45	57	68	79	90	102
39	59106	59218	59329	59439	59550	59660	59770	59879	59988	60097		11	22	33	44	55	66	77	88	99
40	60206	60314	60423	60531	60638	60746	60853	60959	61066	61172		11	21	32	43	54	64	75	86	97
41	61278	61384	61490	61595	61700	61805	61909	62014	62118	62221		11	21	32	42	52	63	73	84	94
42	62325	62428	62531	62634	62737	62839	62941	63043	63144	63246		10	20	31	41	51	61	71	82	92
43	63347	63448	63548	63649	63749	63849	63949	64048	64147	64246		10	20	30	40	50	60	70	80	90
44	64345	64444	64542	64640	64738	64836	64933	65031	65128	65225		10	20	29	39	49	59	68	78	88
45	65321	65418	65514	65610	65706	65801	65896	65992	66087	66181		10	19	29	38	48	57	67	76	86
46	66276	66370	66464	66558	66652	66745	66839	66932	67025	67117		9	19	28	37	47	56	65	75	84
47	67210	67302	67394	67486	67578	67669	67761	67852	67943	68034		9	18	27	37	46	55	64	73	82
48	68124	68215	68305	68395	68485	68574	68664	68753	68842	68931		9	18	27	36	45	53	62	71	80
49	69020	69108	69197	69285	69373	69461	69548	69636	69723	69810		9	18	26	35	44	53	61	70	79
50	69897	69984	70070	70157	70243	70329	70415	70501	70586	70672		9	17	26	34	43	52	60	69	77
51	70757	70842	70927	71012	71096	71181	71265	71349	71433	71517		8	17	25	34	42	51	59	67	76
52	71600	71684	71767	71850	71933	72016	72099	72181	72263	72346		8	17	25	33	42	50	58	66	75
53	72428	72509	72591	72673	72754	72835	72916	72997	73078	73159		8	16	24	32	41	49	57	65	73
54	73239	73320	73400	73480	73560	73640	73719	73799	73878	73957		8	16	24	32	40	48	56	64	72

LOGARITHMS OF NUMBERS

	0	1	2	3	4	5	6	7	8	9	Mean Differences 1	2	3	4	5	6	7	8	9
55	74036	74115	74194	74273	74351	74429	74507	74586	74663	74741	8	16	23	31	39	47	55	62	70
56	74819	74896	74974	75051	75128	75205	75282	75358	75435	75511	8	15	23	31	39	46	54	62	69
57	75587	75664	75740	75815	75891	75967	76042	76118	76193	76268	8	15	23	30	39	45	53	62	68
58	76343	76418	76492	76567	76641	76716	76790	76864	76938	77012	7	15	22	30	37	45	52	59	67
59	77085	77159	77232	77305	77379	77452	77525	77597	77670	77743	7	15	22	29	37	44	51	58	66
60	77815	77887	77960	78032	78104	78176	78247	78319	78390	78462	7	14	22	29	36	43	50	57	65
61	78533	78604	78675	78746	78817	78888	78958	79029	79099	79169	7	14	21	28	35	42	49	56	64
62	79239	79309	79379	79449	79518	79588	79657	79727	79796	79865	7	14	21	28	35	42	49	56	63
63	79934	80003	80072	80140	80209	80277	80346	80414	80482	80550	7	14	21	27	34	41	48	55	62
64	80618	80686	80754	80821	80889	80956	81023	81090	81158	81224	7	13	20	27	34	40	47	54	60
65	81291	81358	81425	81491	81558	81624	81690	81757	81823	81889	7	13	20	26	33	40	46	53	59
66	81954	82020	82086	82151	82217	82282	82347	82413	82478	82543	7	13	20	26	33	39	46	52	59
67	82607	82672	82737	82802	82866	82930	82995	83059	83123	83187	6	13	19	26	32	39	45	52	58
68	83251	83315	83378	83442	83506	83569	83632	83696	83759	83822	6	13	19	25	32	38	44	50	57
69	83885	83948	84011	84073	84136	84198	84261	84323	84386	84448	6	13	19	25	31	38	44	50	56
70	84510	84572	84634	84696	84757	84819	84880	84942	85003	85065	6	12	18	25	31	37	43	49	55
71	85126	85187	85248	85309	85370	85431	85491	85552	85612	85673	6	12	18	24	30	36	42	49	55
72	85733	85794	85854	85914	85974	86034	86094	86153	86213	86273	6	12	18	24	30	36	42	48	54
73	86332	86392	86451	86510	86570	86629	86688	86747	86806	86864	6	12	18	24	30	35	41	47	53
74	86923	86982	87040	87099	87157	87216	87274	87332	87390	87448	6	12	18	23	29	35	41	47	52

	0	1	2	3	4	5	6	7	8	9
75	87506	87564	87622	87679	87737	87795	87852	87910	87967	88024
76	88081	88138	88195	88252	88309	88366	88423	88480	88536	88593
77	88649	88705	88762	88818	88874	88930	88986	89042	89098	89154
78	89209	89265	89321	89376	89432	89487	89542	89597	89653	89708
79	89763	89818	89873	89927	89982	90037	90091	90146	90200	90255
80	90309	90363	90417	90472	90526	90580	90634	90687	90741	90795
81	90849	90902	90956	91009	91062	91116	91169	91222	91275	91328
82	91381	91434	91487	91540	91593	91645	91698	91751	91803	91855
83	91908	91960	92012	92065	92117	92169	92221	92273	92324	92376
84	92428	92480	92531	92583	92634	92686	92737	92788	92840	92891
85	92942	92993	93044	93095	93146	93197	93247	93298	93349	93399
86	93450	93500	93551	93601	93651	93702	93752	93802	93852	93902
87	93952	94002	94052	94101	94151	94201	94250	94300	94349	94399
88	94448	94498	94547	94596	94645	94694	94743	94792	94841	94890
89	94939	94988	95036	95085	95134	95182	95231	95279	95328	95376
90	95424	95472	95521	95569	95617	95665	95713	95761	95809	95856
91	95904	95952	95999	96047	96095	96142	96190	96237	96284	96332
92	96379	96426	96473	96520	96567	96614	96661	96708	96755	96802
93	96848	96895	96942	96988	97035	97081	97128	97174	97220	97267
94	97313	97359	97405	97451	97497	97543	97589	97635	97681	97727
95	97772	97818	97864	97909	97955	98000	98046	98091	98137	98182
96	98227	98272	98318	98363	98408	98453	98498	98543	98588	98632
97	98677	98722	98767	98811	98856	98900	98945	98989	99034	99078
98	99123	99167	99211	99255	99300	99344	99388	99432	99476	99520
99	99564	99607	99651	99695	99739	99782	99826	99870	99913	99957

Proportional Parts

	1	2	3	4	5	6	7	8	9
	6	12	17	23	29	35	40	46	52
	6	11	17	23	29	34	40	46	51
	6	11	17	22	28	34	39	45	50
	6	11	17	22	28	33	38	44	50
	5	11	16	22	27	33	38	44	49
	5	11	16	22	27	32	38	43	49
	5	11	16	21	27	32	37	43	48
	5	11	16	21	26	32	37	42	47
	5	10	16	21	26	31	36	42	47
	5	10	15	21	26	31	36	41	46
	5	10	15	20	25	30	36	41	46
	5	10	15	20	25	30	35	40	45
	5	10	15	20	25	30	35	40	45
	5	10	15	20	25	29	34	39	44
	5	10	15	19	24	29	34	39	44
	5	10	14	19	24	29	34	38	43
	5	9	14	19	24	29	33	38	43
	5	9	14	19	24	28	33	38	42
	5	9	14	19	23	28	33	37	42
	5	9	14	18	23	28	32	37	41
	5	9	14	18	23	27	32	36	41
	5	9	14	18	23	27	32	36	41
	4	9	13	18	22	27	31	36	40
	4	9	13	18	22	26	31	35	40
	4	9	13	17	22	26	30	35	39

UNIVERSITY QUESTIONS

I. A. & I. Sc. Examination Questions

CALCUTTA

1962

1. (a) What is meant by the modulus of a complex number ? Prove that the modulus of the product of two complex numbers is equal to the product of their moduli.

(b) If $z + z_1$ and $\bar{z}, \bar{z_1}$ are both real, where z and z_1 are two complex numbers and $\bar{z}$ and $\bar{z_1}$ are their conjugates, show that *either*, z and z_1 are both real, *or*, $z_1 = \bar{z}$.

2. (a) Solve

$$\frac{x}{2} + \frac{y}{5} = 5 \; ; \quad \frac{2}{x} + \frac{5}{y} = \frac{5}{6}.$$

(b) If x is a real quantity, prove that the expression $\dfrac{x^2 + 2x - 11}{2(x-3)}$ can have all numerical values except such as lie between 2 and 6.

3. (a) Find the number of combinations of n different things taken r at a time.

(b) A committee of 6 is to be chosen from 10 men and 7 women so as to contain at least 3 men and 2 women. In how many different ways can this be done if two particular women refuse to serve on the same committee ?

4. (a) Prove that in the expansion of $(1+x)^n$, where n is a positive integer, the coefficients of terms equidistant from the beginning and end are equal.

(b) The second, third and fourth terms in the expansion of $(x+y)^n$ in descending powers of x (n being a positive integer) are 240, 720 and 1080 respectively ; find x, y and n.

5. (a) Show that $\log_{10} \frac{75}{16} - 2 \log_{10} \frac{5}{9} + \log_{10} \frac{32}{243} = \log_{10} 2$.

(b) If P, Q, R be the pth, qth and rth terms of a geometrical progression, prove that

$$(q-r) \log P + (r-p) \log Q + (p-q) \log R = 0.$$

6. (a) Write down the expansion of $\log_e (1+x)$ in a series of ascending powers of x. Is the expansion valid for all values of x ?

Find the coefficient of x^n in the expansion of $\log_e (1 - 2x + x^2)$ in ascending powers of x, given $|x| < 1$.

(b) Prove that $\dfrac{1}{4} - \dfrac{1}{2 . 4^2} + \dfrac{1}{3 . 4^3} - \dfrac{1}{4 . 4^4} + \cdots\cdots$ to infinity

$$= \frac{1}{5} + \frac{1}{2 . 5^2} + \frac{1}{3 . 5^3} + \frac{1}{4 . 5^4} + \cdots\cdots \text{ to infinity.}$$
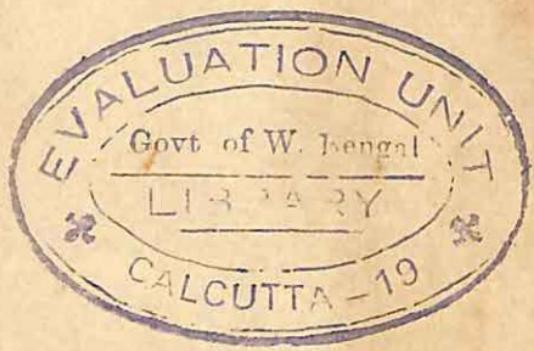
Some University Text-Books of Repute